Department of Theoretical Physics

# The Whittaker model of the center of the quantum group and Hecke algebras

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ABSTRACT

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In 1978 Kostant suggested the *Whittaker model* of the center of the universal enveloping algebra $U(\mathfrak{g})$ of a complex simple Lie algebra $\mathfrak{g}$. An essential role in this construction is played by a non–singular character $\chi$ of the maximal nilpotent subalgebra $\mathfrak{n}_+ \subset \mathfrak{g}$. The main result is that the center of $U(\mathfrak{g})$ is isomorphic to a commutative subalgebra in $U(\mathfrak{b}_-)$, where $\mathfrak{b}_- \subset \mathfrak{g}$ is the opposite Borel subalgebra. This observation is used in the theory of principal series representations of the corresponding Lie group $G$ and in the proof of complete integrability of the quantum Toda lattice.

We show that the Whittaker model introduced by Kostant has a natural homological interpretation in terms of Hecke algebras. Moreover, we introduce a general definition of a Hecke algebra $Hk^*(A, B, \chi)$ associated to the triple of an associative algebra $A$, a subalgebra $B \subset A$ and a character $\chi$ of $B$. In particular, the Whittaker model of the center of $U(\mathfrak{g})$ is identified with $Hk^0(U(\mathfrak{g}), U(\mathfrak{n}_+), \chi)^{opp}$.

The goal of this thesis is to generalize the Kostant's construction to quantum groups. An obvious obstruction is the fact that the subalgebra in $U_h(\mathfrak{g})$ generated by positive root generators (subject to the quantum Serre relations) does not have non–singular characters. In order to overcome this difficulty we introduce a family of new realizations of quantum groups, one for each Coxeter element of the corresponding Weyl group. The modified quantum Serre relations allow for non–singular characters, and we are able to construct the Whittaker model of the center of $U_h(\mathfrak{g})$.

The new Whittaker model is applied to the deformed quantum Toda lattice recently studied by Etingof. We give new proofs of his results which resemble the original Kostant's proofs for the quantum Toda lattice.

Finally, we study the "quasi-classical" limit of the Whittaker model for $U_h(\mathfrak{g})$. A remarkable new result is a cross-section theorem for the action of a complex simple Lie group on itself by conjugations. We are able to prove this theorem for all such Lie groups except for the case of $E_6$! Using the cross–section theorem we establish a relation between the Whittaker model and the set of conjugacy classes of regular elements in the corresponding Lie group $G$.

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# Introduction

In 1978 Kostant suggested the *Whittaker model* of the center of the universal enveloping algebra $U(\mathfrak{g})$ of a complex simple Lie algebra $\mathfrak{g}$. An essential role in this construction is played by a non–singular character $\chi$ of the maximal nilpotent subalgebra $\mathfrak{n}_+ \subset \mathfrak{g}$. The main result is that the center of $U(\mathfrak{g})$ is isomorphic to a commutative subalgebra in $U(\mathfrak{b}_-)$, where $\mathfrak{b}_- \subset \mathfrak{g}$ is the opposite Borel subalgebra. This observation is used in the theory of principal series representations of the corresponding Lie group $G$ and in the proof of complete integrability of the quantum Toda lattice.

We show that the Whittaker model introduced by Kostant has a natural homological interpretation in terms of Hecke algebras. Moreover, following [28] we introduce a general definition of a Hecke algebra $Hk^*(A, B, \chi)$ associated to the triple of an associative algebra $A$, a subalgebra $B \subset A$ and a character $\chi$ of $B$. In particular, the Whittaker model of the center of $U(\mathfrak{g})$ is identified with $Hk^0(U(\mathfrak{g}), U(\mathfrak{n}_+), \chi)^{opp}$.

The goal of this thesis is to generalize the Kostant's construction to quantum groups. An obvious obstruction is the fact that the subalgebra in $U_h(\mathfrak{g})$ generated by positive root generators (subject to the quantum Serre relations) does not have non-singular characters. In order to overcome this difficulty we introduce a family of new realizations of quantum groups, one for each Coxeter element of the corresponding Weyl group (see also [27]). The modified quantum Serre relations allow for non–singular characters, and we are able to construct the Whittaker model of the center of $U_h(\mathfrak{g})$.

The new Whittaker model is applied to the deformed quantum Toda lattice recently studied by Etingof (see [10]). We give new proofs of his results which resemble the original Kostant's proofs for the quantum Toda lattice.

Finally, we study the "quasi-classical" limit of the Whittaker model for $U_h(\mathfrak{g})$. A remarkable new result is a cross-section theorem for the action of a complex simple Lie group on itself by conjugations. We are able to prove this theorem for all such Lie groups except for the case of $E_6$! This theorem is a group counterpart of the cross-section theorem of Kostant (Theorem C, Section 1.3). Using the cross–section theorem we establish a relation between the Whittaker model and the set of conjugacy classes of regular elements in the corresponding Lie group $G$.

The thesis is organized as follows. Chapter 1 contains a review of Kostant's results on the Whittaker model [16], [17]. In order to create a pattern for proofs in the quantum group case we recall most of the Kostant's proofs.

Chapter 2 is devoted to Hecke algebras. It contains the definition of the algebra $Hk^*(A, B, \chi)$ and the interpretation of the Whittaker model as $Hk^0(U(\mathfrak{g}), U(\mathfrak{n}_+), \chi)^{opp}$. The central part of the thesis is Chapter 3. There we describe new realizations of finite-dimensional quantum groups and present the Whittaker model of the center of $U_h(\mathfrak{g})$. Chapter 3 also contains a discussion of the deformed quantum Toda lattice. In Chapter 4 we establish a relation between the Whittaker model and regular elements in algebraic groups. The main result of this Chapter is a cross-section theorem for the action of a complex simple Lie group on itself by conjugations (see also [26] where we prove a modification of this theorem for loop groups).

Many results presented in this thesis for finite-dimensional quantum groups have natural counterparts for affine quantum groups. In order to simplify the presentation we treat only the finite-dimensional case, and refer the reader to the papers [27], [12], [26] for further details on the affine case.

# Contents

# Chapter 1

# Whittaker model

In this chapter we recall the Whittaker model of the center of the universal enveloping algebra $U(\mathfrak{g})$, where $\mathfrak{g}$ is a complex simple Lie algebra.

## 1.1 Notation

Fix the notation used throughout of the text. Let $G$ be a connected simply connected finite–dimensional complex simple Lie group, $\mathfrak{g}$ its Lie algebra. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let $\Delta$ be the set of roots of $(\mathfrak{g}, \mathfrak{h})$. Choose an ordering in the root system. Let $\alpha_i,\ i = 1, \ldots l,\ \ l = rank(\mathfrak{g})$ be the simple roots, $\Delta_+ = \{\beta_1, \ldots, \beta_N\}$ the set of positive roots. Denote by $\rho$ a half of the sum of positive roots, $\rho = \frac{1}{2}\sum_{i=1}^{N}\beta_i$. Let $H_1, \ldots, H_l$ be the set of simple root generators of $\mathfrak{h}$.

Let $a_{ij}$ be the corresponding Cartan matrix. Let $d_1, \ldots, d_l$ be coprime positive integers such that the matrix $b_{ij} = d_i a_{ij}$ is symmetric. There exists a unique non–degenerate invariant symmetric bilinear form $(,)$ on $\mathfrak{g}$ such that $(H_i, H_j) = d_j^{-1}a_{ij}$. It induces an isomorphism of vector spaces $\mathfrak{h} \simeq \mathfrak{h}^*$ under which $\alpha_i \in \mathfrak{h}^*$ corresponds to $d_i H_i \in \mathfrak{h}$. We denote by $\alpha^\vee$ the element of $\mathfrak{h}$ that corresponds to $\alpha \in \mathfrak{h}^*$ under this isomorphism. The induced bilinear form on $\mathfrak{h}^*$ is given by $(\alpha_i, \alpha_j) = b_{ij}$.

Let $W$ be the Weyl group of the root system $\Delta$. $W$ is the subgroup of $GL(\mathfrak{h})$ generated by the fundamental reflections $s_1, \ldots, s_l$,

$$s_i(h) = h - \alpha_i(h)H_i, \quad h \in \mathfrak{h}.$$

The action of $W$ preserves the bilinear form $(,)$ on $\mathfrak{h}$. We denote a representative of $w \in W$ in $G$ by the same letter. For $w \in W, g \in G$ we write

$w(g) = wgw^{-1}$.

Let $\mathfrak{b}_+$ be the positive Borel subalgebra and $\mathfrak{b}_-$ the opposite Borel subalgebra; let $\mathfrak{n}_+ = [\mathfrak{b}_+, \mathfrak{b}_+]$ and $\mathfrak{n}_- = [\mathfrak{b}_-, \mathfrak{b}_-]$ be their nil-radicals. Let $H = \exp \mathfrak{h}, N_+ = \exp \mathfrak{n}_+, N_- = \exp \mathfrak{n}_-, B_+ = HN_+, B_- = HN_-$ be the Cartan subgroup, the maximal unipotent subgroups and the Borel subgroups of $G$ which correspond to the Lie subalgebras $\mathfrak{h}, \mathfrak{n}_+, \mathfrak{n}_-, \mathfrak{b}_+$ and $\mathfrak{b}_-$, respectively.

We identify $\mathfrak{g}$ and its dual by means of the canonical invariant bilinear form. Then the coadjoint action of $G$ on $\mathfrak{g}^*$ is naturally identified with the adjoint one. We also identify ${\mathfrak{n}_+}^* \cong \mathfrak{n}_-,\ {\mathfrak{b}_+}^* \cong \mathfrak{b}_-$.

Let $\mathfrak{g}_\beta$ be the root subspace corresponding to a root $\beta \in \Delta$, $\mathfrak{g}_\beta = \{x \in \mathfrak{g} | [h, x] = \beta(h)x$ for every $h \in \mathfrak{h}\}$. $\mathfrak{g}_\beta \subset \mathfrak{g}$ is a one–dimensional subspace. It is well–known that for $\alpha \neq -\beta$ the root subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to the canonical invariant bilinear form. Moreover $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are non–degenerately paired by this form.

Root vectors $X_\alpha \in \mathfrak{g}_\alpha$ satisfy the following relations:

$$[X_\alpha, X_{-\alpha}] = (X_\alpha, X_{-\alpha})\alpha^\vee.$$

If $V$ is a finite–dimensional complex vector space, $S(V)$ will denote the symmetric algebra over $V$ and $S_k(V)$ denotes the homogeneous subspace of degree k. If $V^*$ is the dual space to $V$ then $S(V^*)$ is regarded as the algebra of polynomial functions on $V$.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and $U_k(\mathfrak{g})$ the standard filtration in $U(\mathfrak{g})$. From the Poincaré–Birkhoff–Witt theorem it follows that the associated graded algebra $GrU(\mathfrak{g})$ is isomorphic to the symmetric algebra $S(\mathfrak{g})$ of the linear space $\mathfrak{g}$.

Equip $S(\mathfrak{g})$ with a Poisson structure as follows. For each $s_k \in S_k(\mathfrak{g})$ choose a representative $u_k \in U_k(\mathfrak{g})$ such that $u_k/U_{k-1}(\mathfrak{g}) = s_k$. We shall denote $s_k = Gru_k$. Given two such elements $s_i$ and $s_j$ with chosen representatives $u_i$ and $u_j$, the commutativity of $S(\mathfrak{g})$ implies that

$$[u_i, u_j] \in U_{i+j-1}(\mathfrak{g}).$$

Define

$$\{s_i, s_j\} = [u_i, u_j]/U_{i+j-2}(\mathfrak{g}). \tag{1.1.1}$$

It is easy to see that this bracket is independent of the choice of representatives $u_i,\ u_j$ and equips $S(\mathfrak{g})$ with the structure of a Poisson algebra, i.e. it is a derivation of the multiplication in $S(\mathfrak{g})$. We refer to the procedure described above as the graded limit.

## 1.2 The Whittaker model

In this section we introduce the Whittaker model of the center of the universal enveloping algebra $U(\mathfrak{g})$. We start by recalling the classical result of Chevalley which describes the structure of the center.

Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. The standard filtration $U_k(\mathfrak{g})$ in $U(\mathfrak{g})$ induces a filtration $Z_k(\mathfrak{g})$ in $Z(\mathfrak{g})$. The following important theorem may be found for instance in [2], Ch.8, §8, no. 3, Corollary 1 and no.5, Theorem 2.

**Theorem (Chevalley)** *One can choose elements $I_k \in Z_{m_k+1}(\mathfrak{g}), \quad k = 1, \ldots l$, where $m_k$ are called the exponents of $\mathfrak{g}$, such that $Z(\mathfrak{g}) = \mathbb{C}[I_1, \ldots, I_l]$ is a polynomial algebra in $l$ generators.*

The adjoint action of $G$ on $\mathfrak{g}$ naturally extends to $S(\mathfrak{g})$. Let $S(\mathfrak{g})^G$ be the algebra of $G$–invariants in $S(\mathfrak{g})$. Clearly, $GrZ(\mathfrak{g}) \cong S(\mathfrak{g})^G$. In particular $S(\mathfrak{g})^G \cong \mathbb{C}[\widehat{I}_1, \ldots, \widehat{I}_l]$, where $\widehat{I}_i = GrI_i, \ i = 1, \ldots, l$. The elements $\widehat{I}_i, \ i = 1, \ldots, l$ are called fundamental invariants.

Following Kostant we shall realize the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ as a subalgebra in $U(\mathfrak{b}_-)$. Let

$$\chi : \mathfrak{n}_+ \to \mathbb{C}$$

be a character of $\mathfrak{n}_+$. Since $\mathfrak{n}_+ = \sum_{i=1}^l \mathbb{C}X_{\alpha_i} \oplus [\mathfrak{n}_+, \mathfrak{n}_+]$ it is clear that $\chi$ is completely determined by the constants $c_i = \chi(X_{\alpha_i}), \ i = 1, \ldots, l$ and $c_i$ are arbitrary. In [16] $\chi$ is called non–singular if $c_i \neq 0$ for all $i$.

Let $f = \sum_{i=1}^l X_{-\alpha_i} \in \mathfrak{n}_-$ be a regular nilpotent element. From the properties of the invariant bilinear form (see Section 1.1) it follows that $(f, [\mathfrak{n}_+, \mathfrak{n}_+]) = 0, \ (f, X_{\alpha_i}) = (X_{-\alpha_i}, X_{\alpha_i})$, and hence the map $x \mapsto (f, x), \ x \in \mathfrak{n}_+$ is a non–singular character of $\mathfrak{n}_+$.

Recall that in our choice of root vectors no normalization was made. But now given a non–singular character $\chi : \mathfrak{n}_+ \to \mathbb{C}$ we will say that $f$ corresponds to $\chi$ in case

$$\chi(X_{\alpha_i}) = (X_{-\alpha_i}, X_{\alpha_i}).$$

Conversely if $\chi$ is non–singular there is a unique choice of $f$ so that $f$ corresponds to $\chi$. In this case $\chi(x) = (f, x)$ for every $x \in \mathfrak{n}_+$.

Naturally, the character $\chi$ extends to a character of the universal enveloping algebra $U(\mathfrak{n}_+)$. Let $U_\chi(\mathfrak{n}_+)$ be the kernel of this extension so that one has a direct sum

$$U(\mathfrak{n}_+) = \mathbb{C} \oplus U_\chi(\mathfrak{n}_+).$$

Since $\mathfrak{g} = \mathfrak{b}_- \oplus \mathfrak{n}_+$ we have a linear isomorphism $U(\mathfrak{g}) = U(\mathfrak{b}_-) \otimes U(\mathfrak{n}_+)$ and hence the direct sum

$$U(\mathfrak{g}) = U(\mathfrak{b}_-) \oplus I_\chi, \tag{1.2.1}$$

where $I_\chi = U(\mathfrak{g})U_\chi(\mathfrak{n}_+)$ is the left–sided ideal generated by $U_\chi(\mathfrak{n}_+)$.

For any $u \in U(\mathfrak{g})$ let $u^\chi \in U(\mathfrak{b}_-)$ be its component in $U(\mathfrak{b}_-)$ relative to the decomposition (1.2.1). Denote by $\rho_\chi$ the linear map

$$\rho_\chi : U(\mathfrak{g}) \to U(\mathfrak{b}_-)$$

given by $\rho_\chi(u) = u^\chi$. Let $W(\mathfrak{b}_-) = \rho_\chi(Z(\mathfrak{g}))$.

**Theorem A ([16], Theorem 2.4.2)** *The map*

$$\rho_\chi : Z(\mathfrak{g}) \to W(\mathfrak{b}_-) \tag{1.2.2}$$

*is an isomorphism of algebras. In particular*

$$W(\mathfrak{b}_-) = \mathbb{C}[I_1^\chi, \dots, I_l^\chi], \quad I_i^\chi = \rho_\chi(I_i), \quad i = 1, \dots, l$$

*is a polynomial algebra in $l$ generators.*

*Proof.* First, we show that the map (1.2.2) is an algebra homomorphism. If $u, v \in Z(\mathfrak{g})$ then $u^\chi v^\chi \in U(\mathfrak{b}_-)$ and

$$uv - u^\chi v^\chi = (u - u^\chi)v + u^\chi(v - v^\chi).$$

Since $(u - u^\chi)v = v(u - u^\chi)$ the r.h.s. of the last equality is an element of $I_\chi$. This proves $u^\chi v^\chi = (uv)^\chi$.

By definition the map (1.2.2) is surjective. We have to prove that it is injective. Let $U(\mathfrak{g})^\mathfrak{h}$ be the centralizer of $\mathfrak{h}$ in $U(\mathfrak{g})$. Clearly $Z(\mathfrak{g}) \subseteq U(\mathfrak{g})^\mathfrak{h}$. From the Poincaré–Birkhoff–Witt theorem it follows that every element $z \in U(\mathfrak{g})^\mathfrak{h}$ may be uniquely written as

$$z = \sum_{p,q \in \mathbb{N}^N, <p>=<q>} X_{-\beta_1}^{p_1} \dots X_{-\beta_N}^{p_N} \varphi_{p,q} X_{\beta_1}^{q_1} \dots X_{\beta_N}^{q_N},$$

where $< p >= \sum_{i=1}^r p_i \beta_i \in \mathfrak{h}^*$ and $\varphi_{p,q} \in U(\mathfrak{h})$.

Now recall that $\chi(X_{\beta_i}) = 0$ if $\beta_i$ is not a simple root, and we easily obtain

$$\rho_\chi(z) = \sum_{p,q \in \mathbb{N}^l, <p>=<q>\neq 0} X_{-\alpha_{k_1}}^{p_{j_1}} \dots X_{-\alpha_{k_l}}^{p_{j_l}} \varphi_{p,q} \prod_{i=1}^{l} c_{k_i}^{q_{j_i}} + \varphi_{0,0}.$$

Let $z \in Z(\mathfrak{g})$. One knows that the map

$$Z(\mathfrak{g}) \to U(\mathfrak{h}), \quad z \mapsto \varphi_{0,0},$$

called the Harich-Chandra homomorphism, is injective (see (c), p. 232 in [7]). It follows that the map (1.2.2) is also injective.

**Definition A** *The algebra $W(\mathfrak{b}_-)$ is called the Whittaker model of $Z(\mathfrak{g})$.*

Next we equip $U(\mathfrak{b}_-)$ with a structure of a left $U(\mathfrak{n}_+)$ module in such a way that $W(\mathfrak{b}_-)$ is realized as the space of invariants with respect to this action.

Let $Y_\chi$ be the left $U(\mathfrak{g})$ module defined by

$$Y_\chi = U(\mathfrak{g}) \otimes_{U(\mathfrak{n}_+)} \mathbb{C}_\chi,$$

where $\mathbb{C}_\chi$ denotes the 1–dimensional $U(\mathfrak{n}_+)$–module defined by $\chi$. Obviously $Y_\chi$ is just the quotient module $U(\mathfrak{g})/I_\chi$. From (1.2.1) it follows that the map

$$U(\mathfrak{b}_-) \to Y_\chi; \quad v \mapsto v \otimes 1 \tag{1.2.3}$$

is a linear isomorphism.

It is convenient to carry the module structure of $Y_\chi$ to $U(\mathfrak{b}_-)$. For $u \in U(\mathfrak{g})$, $v \in U(\mathfrak{b}_-)$ the induced action $u \circ v$ has the form

$$u \circ v = (uv)^\chi. \tag{1.2.4}$$

The restriction of this action to $U(\mathfrak{n}_+)$ may be changed by tensoring with 1–dimensional $U(\mathfrak{n}_+)$–module defined by $-\chi$. That is $U(\mathfrak{b}_-)$ becomes an $U(\mathfrak{n}_+)$ module where if $x \in U(\mathfrak{n}_+)$, $v \in U(\mathfrak{b}_-)$ one puts

$$x \cdot v = x \circ v - \chi(x)v. \tag{1.2.5}$$

**Lemma A ([16], Lemma 2.6.1.)** *Let $v \in U(\mathfrak{b}_-)$ and $x \in U(\mathfrak{n}_+)$. Then*

$$x \cdot v = [x, v]^\chi.$$

*Proof.* By definition $x \cdot v = (xv)^\chi - \chi(x)v$. Then we have $xv = [x, v] + vx$ and hence $x \cdot v = ([x, v])^\chi + (vx)^\chi - \chi(x)v$. But clearly $(vx)^\chi = v\chi(x)$. Thus $x \cdot v = ([x, v])^\chi$.

The action (1.2.5) may be lifted to an action of the unipotent group $N_+$. Consider the space $U(\mathfrak{b}_-)^{N_+}$ of $N_+$ invariants in $U(\mathfrak{b}_-)$ with respect to this action. Clearly, $W(\mathfrak{b}_-) \subseteq U(\mathfrak{b}_-)^{N_+}$ .

**Theorem B ([16], Theorems 2.4.1, 2.6)** *Suppose that the character $\chi$ is non–singular. Then the space of $N_+$ invariants in $U(\mathfrak{b}_-)$ with respect to the action (1.2.5) is isomorphic to $W(\mathfrak{b}_-)$, i.e.*

$$U(\mathfrak{b}_-)^{N_+} \cong W(\mathfrak{b}_-). \tag{1.2.6}$$

We shall prove Theorem B in the next section.

## 1.3 Geometric approach to the Whittaker model

In this section we establish a relation between the Whittaker model and the geometry of the adjoint action of the corresponding Lie group.

Denote the character of $S(\mathfrak{n}_+)$ that equals to $\chi(x)$ for every $x \in \mathfrak{n}_+$ by the same letter. Similarly to (1.2.1) we have the following decomposition for $S(\mathfrak{g})$:

$$S(\mathfrak{g}) = S(\mathfrak{b}_-) \oplus I_\chi^0,$$

where $I_\chi^0$ is the ideal in $S(\mathfrak{g})$ generated by the kernel of $\chi$. For any $s \in S(\mathfrak{g})$ let $s^\chi$ be its component in $S(\mathfrak{b}_-)$ relative to this decomposition.

Now using Lemma A we define the graded limit of the action (1.2.5). For $x \in \mathfrak{n}_+$ and $s \in S(\mathfrak{b}_-)$ we put

$$x \cdot s = (\{x, s\})^\chi. \tag{1.3.1}$$

This action may be lifted to an action of the unipotent group $N_+$ on $S(\mathfrak{b}_-)$. For $a \in N_+,\ s \in S(\mathfrak{b}_-)$ this action is given by

$$a \cdot s = (\mathrm{Ad}(a)(s))^\chi.$$

Observe that $S(\mathfrak{b}_-)$ is naturally identified with the algebra of polynomial functions on $\mathfrak{b}_+$. We shall describe the space of invariants $S(\mathfrak{b}_-)^{N_+}$ using the induced action of $N_+$ on $\mathfrak{b}_+$.

To calculate this action it suffices to consider the restriction of the action (1.3.1) to linear functions. Let $s \in \mathfrak{b}_-$ be such a function. Then for $a \in N_+$

$$a \cdot s = (\mathrm{Ad}(a)(s))^\chi = P_{\mathfrak{b}_-}(\mathrm{Ad}(a)(s)) + \chi(P_{\mathfrak{n}_+}(\mathrm{Ad}(a)(s))),$$

where $P_{\mathfrak{b}_-}$ and $P_{\mathfrak{n}_+}$ are the projection operators onto $\mathfrak{b}_-$ and $\mathfrak{n}_+$, respectively, in the direct sum $\mathfrak{g} = \mathfrak{b}_- + \mathfrak{n}_+$. By the definition of the induced action we have

$$a \cdot s(s') = s(a^{-1} \cdot s'), \text{ for every } s' \in \mathfrak{b}_+.$$

On the other hand

$$a \cdot s(s') = (P_{\mathfrak{b}_-}(\mathrm{Ad}(a)(s)), s') + \chi(P_{\mathfrak{n}_+}(\mathrm{Ad}(a)(s))) = \\ (\mathrm{Ad}(a)(s), s') + (\mathrm{Ad}(a)(s), f),$$

where $f \in \mathfrak{n}_-$ corresponds to $\chi$. Since the canonical bilinear form $(,)$ is Ad-invariant the last formula may be rewritten as:

$$a \cdot s(s') = (s, \mathrm{Ad}(a)^{-1}(s' + f)) = s(P_{\mathfrak{b}_+}(\mathrm{Ad}(a)^{-1}(s' + f))),$$

where $P_{\mathfrak{b}_+}$ is the projector onto $\mathfrak{b}_+$ in the direct sum $\mathfrak{g} = \mathfrak{b}_+ + \mathfrak{n}_-$.

Finally observe that the subspace $f + \mathfrak{b}_+$ is stable under the adjoint action of $N_+$. Therefore $P_{\mathfrak{b}_+}(\mathrm{Ad}(a)^{-1}(s' + f)) = \mathrm{Ad}(a)^{-1}(s' + f) - f$, and the induced action of $N_+$ on $\mathfrak{b}_+$ takes the form:

$$a \cdot s' = \mathrm{Ad}(a)(s' + f) - f. \tag{1.3.2}$$

Now the algebra of invariants $S(\mathfrak{b}_-)^{N_+}$ may be identified with a certain subalgebra in the algebra of functions on the quotient $\mathfrak{b}_+/N_+$. The space $\mathfrak{b}_+/N_+$ has a nice geometric description.

Observe that $[f, \mathfrak{n}_+] \subset \mathfrak{b}_+$. Moreover, $[f, \mathfrak{n}_+]$ is an $\mathrm{ad}_\rho$ stable subspace of $\mathfrak{b}_+$. Since $\rho$ is a semi–simple element there exists an $\mathrm{ad}_\rho$ invariant stable subspace $\mathfrak{s} \subseteq \mathfrak{b}_+$ such that $\mathfrak{b}_+ = \mathfrak{s} + [f, \mathfrak{n}_+]$ is a direct sum. By Theorem 8 and Remark 19' in [17] $\mathfrak{s}$ is an $l$–dimensional subspace in $\mathfrak{n}_+$.

**Theorem C ([16], Theorem 1.2 )** *The map*

$$N_+ \times \mathfrak{s} \to \mathfrak{b}_+$$

*given by* $(a, x) \mapsto a \cdot x$ *is an isomorphism of affine varieties. Therefore the quotient space* $\mathfrak{b}_+/N_+$ *is isomorphic to* $\mathfrak{s}$.

The linear space $\mathfrak{s}$ naturally appears in the study of regular elements in $\mathfrak{g}$. Recall that an element of $\mathfrak{g}$ is called regular if its centralizer in $\mathfrak{g}$ is of minimal possible dimension. Let $R$ be the set of regular elements in $\mathfrak{g}$. Clearly, $R$ is stable under the adjoint action of $G$ and in fact $R$ is the union of all $G$ orbits in $\mathfrak{g}$ of maximal dimension.

**Theorem D ([16], Theorem 1.1; [17], Theorem 8)** *The affine space $f+\mathfrak{s}$ is contained in $R$ and is a cross–section for the action of $G$ on $R$. That is every $G$–orbit in $\mathfrak{g}$ of maximal dimension intersects $f+\mathfrak{s}$ in one and only one point.*

*Let $\widehat{I}_1,\ldots,\widehat{I}_l\in S(\mathfrak{g})^G$ be the fundamental invariants. $\widehat{I}_1,\ldots,\widehat{I}_l$ may be viewed as polynomial functions on $\mathfrak{g}^*\cong\mathfrak{g}$. The restrictions of these functions to $f+\mathfrak{s}$ define a global coordinate system on $\mathfrak{s}$.*

**Theorem E ([16], Theorem 1.3)** *For any $\widehat{I}\in S(\mathfrak{g})^G$ one has $\widehat{I}^\chi\in S(\mathfrak{b}_-)^{N_+}$. Furthermore the map*

$$S(\mathfrak{g})^G\rightarrow S(\mathfrak{b}_-)^{N_+},\quad \widehat{I}\mapsto\widehat{I}^\chi \tag{1.3.3}$$

*is an algebra isomorphism. In particular*

$$S(\mathfrak{b}_-)^N=\mathbb{C}[\widehat{I}_1^\chi,\ldots,\widehat{I}_l^\chi]$$

*is a polynomial algebra in $l$ generators.*

*Proof.* First observe that elements of $S(\mathfrak{g})$ may be viewed as polynomial functions on $\mathfrak{g}^*\cong\mathfrak{g}$. Note also that the ideal $I_\chi^0$ is generated by the elements $x-(x,f),\ x\in\mathfrak{n}_+$. Therefore $I_\chi^0$ is the ideal of polynomial functions vanishing on the subspace $f+\mathfrak{b}_+$ and so for every $\widehat{I}\in S(\mathfrak{g})$ $\widehat{I}^\chi$ may be regarded as the restriction of the function $\widehat{I}$ to the subspace $f+\mathfrak{b}_+$.

For $\widehat{I}\in S(\mathfrak{g})^G,\ s'\in\mathfrak{b}_+$ and $a\in N_+$ one has $\widehat{I}(\mathrm{Ad}(a)(f+s'))=\widehat{I}^\chi(a\cdot s')$. Since $\widehat{I}(\mathrm{Ad}(a)(f+s'))=\widehat{I}(f+s')$ it follows that $\widehat{I}^\chi\in S(\mathfrak{b}_-)^{N_+}$.

By Theorem C the map

$$S(\mathfrak{b}_-)^{N_+}\rightarrow S(\mathfrak{s}^*)$$

given by the restriction $v\mapsto v|_{\mathfrak{s}}$ is an algebra isomorphism. Now by Theorem D the restrictions of the functions $\widehat{I}_i^\chi,\ i=1,\ldots,l$ to $\mathfrak{s}$ are a coordinate system. Therefore (1.3.3) is an isomorphism.

*Proof of Theorem B.* First observe that elements $\widehat{I}_i^\chi,\ i=1,\ldots,l$ are the graded limits of the elements $I_i^\chi\in U(\mathfrak{b}_-)^{N_+},\ i=1,\ldots,l$. Therefore $GrW(\mathfrak{b}_-)=S(\mathfrak{b}_-)^{N_+}$. Recall that $W(\mathfrak{b}_-)\subseteq U(\mathfrak{b}_-)^{N_+}$ is a linear subspace.

Let $J\in U(\mathfrak{b}_-)^{N_+}\cap U_{k_1}(\mathfrak{b}_-)$ be an invariant element. Clearly, $GrJ\in S(\mathfrak{b}_-)^{N_+}$. Since $GrW(\mathfrak{b}_-)=S(\mathfrak{b}_-)^{N_+}$ one can find elements $I_1\in W(\mathfrak{b}_-)\cap U_{k_1}(\mathfrak{b}_-)$ and $J_1\in U(\mathfrak{b}_-)^{N_+}\cap U_{k_2}(\mathfrak{b}_-),\ \ k_2<k_1$ such that

$$J-I_1=J_1.$$

Applying the same procedure to $J_1$ we obtain elements $J_2 \in U(\mathfrak{b}_-)^{N_+} \cap U_{k_3}(\mathfrak{b}_-),\ k_3 < k_2,\ \ I_2 \in W(\mathfrak{b}_-) \cap U_{k_2}(\mathfrak{b}_-)$ such that

$$J_1 - I_2 = J_2.$$

We can continue this process. Since the standard filtration in $U(\mathfrak{b}_-)$ is bounded below we finally obtain that for some $i$ $J_i - I_i = c \in \mathbb{C}$. By construction the element $J$ is represented as $J = \sum_{j=1}^{i} I_j + c,\ I_j \in W(\mathfrak{b}_-) \cap U_{k_j}(\mathfrak{b}_-)$. Therefore $J \in W(\mathfrak{b}_-)$. This concludes the proof.

Now we make an important remark.

**Remark A** Observe that the space $U(\mathfrak{b}_-)^{N_+}$ may be interpreted as the zeroth cohomology space of the $U(\mathfrak{n}_+)$ module $Y_\chi$, where $U(\mathfrak{n}_+)$ is augmented by $\chi$. Indeed, for every associative algebra $B$ equipped with character $\chi$ and for every left $B$–module $V$ the cohomology module $H^*(V)$ is defined as the cohomology space of the complex (see [4])

$$\mathrm{Hom}_B(X, V), \tag{1.3.4}$$

where $X$ is a projective resolution of the one–dimensional $B$–module $\mathbb{C}_\chi$ defined by $\chi$. In homological algebra $\chi$ is called an augmentation of $B$. It is well–known that the graded vector space $H^*(V)$ does not depend on the resolution $X$ and the zeroth cohomology space $H^0(V)$ is isomorphic to the space of invariants $\mathrm{Hom}_B(\mathbb{C}_\chi, V)$ (see [4]). Using the map

$$\mathrm{Hom}_B(\mathbb{C}_\chi, V) \to V; \quad \hat{v} \mapsto \hat{v}(1) = v$$

this space may be identified with the subspace in $V$ spanned by elements $v \in V$ such that $bv = \chi(b)v$ for every $b \in B$, i.e.

$$H^0(V) = \mathrm{Hom}_B(\mathbb{C}_\chi, V) = \{v \in V : bv = \chi(b)v \text{ for every } b \in B\}.$$

Now for $B = U(\mathfrak{n}_+)$, $\chi$ as in Theorem B and $V = Y_\chi$ we have $H^0(Y_\chi) = \{v \in Y_\chi : xv = \chi(x)v \text{ for every } x \in U(\mathfrak{n}_+)\}$. From (1.2.5) and (1.2.3) it follows that $H^0(Y_\chi) = U(\mathfrak{b}_-)^{N_+}$.

Now recall that by Theorem B there exists a linear isomorphism $W(\mathfrak{b}_-) \cong U(\mathfrak{b}_-)^{N_+}$. Therefore the associative algebra $W(\mathfrak{b}_-)$ is isomorphic to $H^0(Y_\chi)$ as a linear space. In Section 2.6 we show that the multiplicative structure of $W(\mathfrak{b}_-)$ naturally appears in the context of homological algebra.

# Chapter 2

# Hecke algebras

In this section we give a homological definition of Hecke algebras (see Section 2.2). Let $K$ be a ring with unit, $A$ an associative algebra over $K$, and $B$ a subalgebra of $A$ with augmentation, that is, a $K$–algebra homomorphism $\varepsilon : B \to K$. The Hecke algebra $Hk^*(A, B, \varepsilon)$ of the triple $(A, B, \varepsilon)$ is a natural generalization of the algebra $\mathrm{Hom}_A(A \otimes_B K, A \otimes_B K)$. For every left $A$ module $V$ and every right $A$ module $W$ the algebra $Hk^*(A, B, \varepsilon)$ acts in both the cohomology space $H^*(B, V)$ and the homology space $H_*(B, W)$ of $V$ and $W$ as $B$–modules. Hecke algebras are also closely related to the quantum BRST cohomology (see [19]).

To define Hecke algebras we study complexes of $A$–endomorphisms of graded left $A$ modules. Let $X$ be such a complex, $\mathrm{End}_A(X)$ be the corresponding complex of endomorphisms. Our main observation is that the natural multiplication in $\mathrm{End}_A(X)$ given by composition of endomorphisms induces a multiplicative structure on the cohomology space $H^*(\mathrm{End}_A(X))$. Furthermore, the associative algebra $H^*(\mathrm{End}_A(X))$ only depends on the homotopy class of the complex $X$.

As an application of our construction we show that the Whittaker model $W(\mathfrak{b}_-)$ is the zeroth graded component of the Hecke algebra of the triple $(U(\mathfrak{g}), U(\mathfrak{n}), \chi)$.

The exposition in this chapter follows [28].

## 2.1 Endomorphisms of complexes

Let $A$ be an associative ring with unit, $X$ a graded complex of left $A$ modules equipped with a differential $d$ of degree $-1$. Recall the definition of the

complex $Y = \mathrm{End}_A(X)$ [21].

By definition $Y$ is a $\mathbb{Z}$–graded complex

$$Y = \bigoplus_{n=-\infty}^{\infty} Y^n$$

with graded components defined as

$$Y^n = \prod_{p+q=n} Y^{p,q},$$

where

$$Y^{p,q} = \mathrm{Hom}_A(X^p, X^{-q}).$$

Clearly $Y$ is closed with respect to the multiplication given by composition of endomorphisms. Thus it is a graded associative algebra.

We introduce a differential on $Y$ of degree $+1$ as follows:

$$\begin{gathered}(\mathbf{d}f)^{p,q} = (-1)^{p+q} f^{p-1,q} \circ d + d \circ f^{p,q-1}, \\ f = \{f^{p,q}\}, f^{p,q} \in Y^{p,q},\end{gathered}$$

where $d$ is the differential of $X$. If $f$ is homogeneous then

$$\mathbf{d}f = d \circ f - (-1)^{\deg(f)} f \circ d. \tag{2.1.1}$$

So that $\mathbf{d}$ is the supercommutator by $d$.

We shall consider also the partial differentials $d'$ and $d''$ on Y :

$$\begin{gathered}\text{for } f \in Y^{p,q} \\ (d'f)(x) = (-1)^{p+q+1} f(dx), \quad x \in X^{p+1}; \\ (d''f)(x) = df(x), \qquad\qquad x \in X^p.\end{gathered} \tag{2.1.2}$$

It is easy to check that

$$d'^2 = d''^2 = d'd'' + d''d' = 0$$

These conditions ensure that $\mathbf{d}^2 = 0$.

The following property of $\mathbf{d}$ is crucial for the subsequent considerations.

**Lemma 2.1.1** $\mathbf{d}$ *is a superderivation of* $Y$*.*

*Proof.* Let $f$ and $g$ be homogeneous elements of $Y$. Then $\deg(fg) = \deg(f) + \deg(g)$ and (2.1.1) yields:

$$\begin{aligned}
&\mathbf{d}(fg) = d \circ fg - (-1)^{\deg(f)+\deg(g)} fg \circ d = \\
&d \circ fg - (-1)^{\deg(f)} f \circ d \circ g + (-1)^{\deg(f)} f \circ d \circ g - (-1)^{\deg(f)+\deg(g)} fg \circ d = \\
&(\mathbf{d}f)g + (-1)^{\deg(f)} f(\mathbf{d}g).
\end{aligned}$$

This completes the proof.

The most important consequence of the lemma is

**Proposition 2.1.2** *The homology space $H^*(Y)$ inherits a multiplicative structure from Y. Thus $H^*(Y)$ is a graded associative algebra.*

*Proof.* First, the product of two cocycles is a cocycle. For if $f$ and $g$ are homogeneous and $\mathbf{d}f = \mathbf{d}g = 0$ then

$$\mathbf{d}(fg) = (\mathbf{d}f)g + (-1)^{\deg(f)} f(\mathbf{d}g) = 0.$$

Now we have to show that the product of homology classes is well–defined. It suffices to verify that the product of a homogeneous cocycle with a homogeneous coboundary is cohomologous to zero. For instance consider the product $f\mathbf{d}h$. Equation (2.1.1) gives

$$\begin{aligned}
f\mathbf{d}h = f \circ (d \circ h - (-1)^{\deg(h)} h \circ d) = \\
(-1)^{\deg(f)} d \circ f \circ h - (-1)^{\deg(h)} f \circ h \circ d = (-1)^{\deg(f)} \mathbf{d}(fh).
\end{aligned} \tag{2.1.3}$$

This completes the proof.

One of the principal statements of homological algebra says that homotopically equivalent complexes have the same homology. In particular the vector space $H^*(Y)$ depends only on the homotopy class of the complex $X$. It turns out that the same is true for the algebraic structure of $H^*(Y)$. Indeed we have the following

**Theorem 2.1.3** *Let $X, X'$ be two homotopically equivalent graded complexes of left $A$–modules. Then*

$$H^*(Y) \cong H^*(Y')$$

*as graded associative algebras.*

*Proof.* Let $F : X \to X', F' : X' \to X$ be two maps between the complexes such that

$$\begin{aligned}
F'F - \mathrm{id}_X = d_X s + s d_X, \qquad & s : X \to X, \qquad s \in Y^{-1}, \\
FF' - \mathrm{id}_{X'} = d_{X'} s' + s' d_{X'}, \quad & s' : X' \to X', \quad s' \in Y'^{-1}.
\end{aligned}$$

Consider the induced mappings of the complexes $Y,\ \ Y'$:

$$
\begin{gathered}
FF'^{*}: Y \to Y', \\
FF'^{*}f = F \circ f \circ F', f \in Y; \\
F'F^{*}: Y' \to Y, \\
F'F^{*}g = F' \circ g \circ F, f \in Y'.
\end{gathered}
$$

Their compositions are homotopic to the identity maps of $Y$ and $Y'$ (see Chap. 4, [4] for a general statement about equivalences of functors). But this means that $FF'^{*}$ is inverse to $F'F^{*}$ when restricted to homology. Thus $H^{*}(Y)$ is isomorphic to $H^{*}(Y')$ as a vector space. We have to show that the restrictions of $FF'^{*}$ and $F'F^{*}$ to the homologies are homomorphisms of algebras.

Let $f$ and $g$ be homogeneous elements of $Y$ and $\mathbf{d}_X f = \mathbf{d}_X g = 0$. By the definition of the induced maps we have

$$FF'^{*}(fg) = F \circ fg \circ F'.$$

On the other hand

$$
\begin{aligned}
FF'^{*}(f)FF'^{*}(g) = F \circ f \circ F'F \circ g \circ F' = \\
F \circ f(\mathrm{id}_X + d_X s + s d_X) g \circ F'.
\end{aligned} \tag{2.1.4}
$$

Now recall that $f$ and $g$ are cocycles in $Y$. By (2.1.1) they supercommute with $d_X$:

$$d_X \circ f = (-1)^{\deg(f)} f \circ d_X. \tag{2.1.5}$$

Using (2.1.5) and the fact that $F$ and $F'$ are morphisms of complexes we can rewrite (2.1.4) as follows:

$$
\begin{aligned}
F \circ f(\mathrm{id}_X + d_X s + s d_X) g \circ F' = F \circ fg \circ F' + \\
+(-1)^{\deg(f)} d_{X'} \circ F \circ fsg \circ F' + (-1)^{\deg(g)} F \circ fsg \circ F' \circ d_{X'} = \\
= F \circ fg \circ F' + (-1)^{\deg(f)} \mathbf{d}_{X'}(F \circ fsg \circ F').
\end{aligned} \tag{2.1.6}
$$

Finally observe that by (2.1.6), $FF'^{*}(fg)$ and $FF'^{*}(f)FF'^{*}(g)$ belong to the same homology class in $H^{*}(Y')$. This completes the proof.

## 2.2 Hecke algebras

Let $A$ be an associative algebra over a ring $K$ with unit, and $B$ a subalgebra of $A$ with augmentation, that is, a $K$–algebra homomorphism $\varepsilon : B \to K$.

Let $X$ be a projective resolution of the left $B$–module $K$ defined by $\varepsilon$. Since $X$ is a complex of left $B$–modules, the space $A \otimes_B X$ is also a differential complex. Observe that this complex has the natural structure of a left $A$–module. Therefore we can apply Proposition 2.1.2 to define a graded associative algebra

$$Hk^*(A, B, \varepsilon) = H^*(\mathrm{End}_A(A \otimes_B X)).$$

Note that all $B$–projective resolutions of $K$ are homotopically equivalent and so the complexes $A \otimes_B X$ are homotopically equivalent for different resolutions $X$. Hence by Theorem 2.1.3 the associative algebra $Hk^*(A, B, \varepsilon)$ does not depend on the resolution $X$. We shall call it the *Hecke algebra* of the the triple $(A, B, \varepsilon)$.

Now consider $A$ as a left $A$–module and a right $B$–module via multiplication. In this way $A$ becomes a left $A \otimes B^{opp}$–module. Let $X'$ be a projective resolution of this module. The complex $X' \otimes_B K$, where the $B$ module structure on $K$ is defined by $\varepsilon$, is a left $A$–module. Therefore one can define an associative algebra

$$\widehat{Hk}^*(A, B, \varepsilon) = H^*(\mathrm{End}_A(X' \otimes_B K))$$

independent of the resolution $X'$.

**Proposition 2.2.1** *$Hk^*(A, B, \varepsilon)$ is isomorphic to $\widehat{Hk}^*(A, B, \varepsilon)$ as a graded associative algebra.*

*Proof.* We shall use the standard bar resolutions for computing $\widehat{Hk}^*(A, B, \varepsilon)$ and $Hk^*(A, B, \varepsilon)$ [21], [4]. Consider the complex $B \otimes T(I(B)) \otimes B$, where $I(B) = B/K$ and $T$ denotes the tensor algebra of the vector space. Elements of $B \otimes T(I(B)) \otimes B$ are usually written as $a[a_1, \ldots, a_s]a'$. The differential is given by

$$da[a_1, \ldots, a_s]a' = aa_1[a_2, \ldots, a_s]a' + \sum_{k=1}^{s-1}(-1)^k a[a_1, \ldots, a_k a_{k+1}, \ldots, a_s]a' + (-1)^s a[a_1, \ldots, a_{s-1}]a_s a'. \quad (2.2.1)$$

Then $B \otimes T(I(B)) \otimes B \otimes_B K = B \otimes T(I(B)) \otimes K$ is a free resolution of the left $B$–module $K$. And $A \otimes_B B \otimes T(I(B)) \otimes B = A \otimes T(I(B)) \otimes B$ is a free resolution of $A$ as a right $B$–module. The complex $A \otimes T(I(B)) \otimes B$ is also a free left $A$–module via left multiplication by elements of $A$. Hence this is an $A \otimes B^{opp}$– free resolution of $A$.

Thus the complex $\mathrm{End}_A(A \otimes_B B \otimes T(I(B)) \otimes K) = \mathrm{End}_A(A \otimes T(I(B)) \otimes K)$ for the computation of $Hk^*(A, B, \varepsilon)$ is canonically isomorphic to the complex $\mathrm{End}_A(A \otimes T(I(B)) \otimes B \otimes_B K) = \mathrm{End}_A(A \otimes T(I(B)) \otimes K)$ for the computation of $\widehat{Hk}^*(A, B, \varepsilon)$. This establishes the isomorphism of the algebras.

## 2.3 Action in homology and cohomology spaces

Recall that for every left $B$–module $V$ the cohomology modules are defined to be

$$H^*(B, V) = \mathrm{Ext}^*_B(K, V) = H^*(\mathrm{Hom}_B(X, V)), \tag{2.3.1}$$

where $X$ is a projective resolution of $K$. On the other hand for every right $B$–module $W$ one can define the homology modules

$$H_*(B, W) = \mathrm{Tor}^B_*(W, K) = H_*(W \otimes_B X). \tag{2.3.2}$$

Now observe that for every left $A$–module $V$ the complex in (2.3.1) for calculating its cohomology as a right $B$–module may be represented as follows:

$$\mathrm{Hom}_B(X, V) = \mathrm{Hom}_A(A \otimes_B X, V). \tag{2.3.3}$$

Endow the space $\mathrm{Hom}_A(A \otimes_B X, V)$ with a right $\mathrm{End}_A(A \otimes_B X)$–action:

$$\begin{gathered} \mathrm{Hom}_A(A \otimes_B X, V) \times \mathrm{End}_A(A \otimes_B X) \to \mathrm{Hom}_A(A \otimes_B X, V), \\ \varphi \times f \mapsto \varphi \circ f, \\ \varphi \in \mathrm{Hom}_A(A \otimes_B X, V), f \in \mathrm{End}_A(A \otimes_B X). \end{gathered} \tag{2.3.4}$$

This action is well–defined since $f$ commutes with the left $A$–action. Clearly this action respects the gradings, i.e., it is an action of the graded associative algebra on the graded module.

**Proposition 2.3.1** *For every left $A$ module $V$ the action (2.3.4) gives rise to a right action*

$$\begin{gathered} H^*(B, V) \times Hk^*(A, B, \varepsilon) \to H^*(B, V), \\ H^n(B, V) \times Hk^m(A, B, \varepsilon) \to H^{n+m}(B, V). \end{gathered} \tag{2.3.5}$$

*Proof.* Let $\varphi \in \mathrm{Hom}_A(A \otimes_B X, V)$ and $d\varphi = \varphi \circ d = 0$. Let also $f \in \mathrm{End}_A(A \otimes_B X)$ be a homogeneous cocycle. By (2.1.5) $\varphi \circ f$ is a cocycle in $\mathrm{Hom}_A(A \otimes_B X, V)$. Indeed

$$d(\varphi \circ f) = \varphi \circ f \circ d = (-1)^{\deg(f)} \varphi \circ d \circ f = 0.$$

Next we need to show that the action does not depend on the choice of the representative $f$ in the homology class $[f]$, that is $\varphi \circ \mathbf{d}g$ is homologous to zero for every homogeneous $g \in \mathrm{End}_A(A \otimes_B X)$. This is a direct consequence of the definitions:

$$\varphi \circ \mathbf{d}g = \varphi \circ (d \circ g - (-1)^{\deg(g)} g \circ d) = -(-1)^{\deg(g)} d(\varphi \circ g),$$

since $\varphi \circ d = 0$.

Finally let us check that the action is independent of the representative in the homology class $[\varphi]$. For $\psi \in \mathrm{Hom}_A(A \otimes_B X, V)$ $d\psi \circ f$ is always homologous to zero:

$$d\psi \circ f = \psi \circ d \circ f = (-1)^{\deg(f)} \psi \circ f \circ d = (-1)^{\deg(f)} d(\psi \circ f).$$

This concludes the proof.

Similarly for every right $A$–module $W$ one can equip the homology module $H_*(B, W)$ with a structure of a left $Hk^*(A, B, \varepsilon)$–module. First the complex $W \otimes_B X = W \otimes_A A \otimes_B X$ has the natural structure of a left $\mathrm{End}_A(A \otimes_B X)$–module:

$$\begin{aligned} &\mathrm{End}_A(A \otimes_B X) \times W \otimes_A A \otimes_B X \to W \otimes_A A \otimes_B X, \\ &f \times w \otimes x \mapsto w \otimes f(x), \\ &w \otimes x \in W \otimes_A (A \otimes_B X), f \in \mathrm{End}_A(A \otimes_B X). \end{aligned} \tag{2.3.6}$$

Observe that according to the convention of Section 2.1 elements of $\mathrm{End}^n_A(A \otimes_B X)$ have degree -n as operators in the graded space $W \otimes_A A \otimes_B X$:

$$\mathrm{End}^n_A(A \otimes_B X) \times W \otimes_A A \otimes_B X_m \to W \otimes_A A \otimes_B X_{m-n}.$$

The following assertion is an analogue of Proposition 2.3.1 for homology.

**Proposition 2.3.2** *For every right $A$ module $W$ the action (2.3.6) gives rise to a left action*

$$\begin{aligned} Hk(A, B, \varepsilon)^* \times H_*(B, W) &\to H_*(B, W), \\ Hk(A, B, \varepsilon)^n \times H_m(B, W) &\to H_{m-n}(B, W). \end{aligned} \tag{2.3.7}$$

## 2.4 Structure of the Hecke algebras

In this section we investigate the Hecke algebras under some technical assumptions. The main theorem here is

**Theorem 2.4.1** *Assume that*

$$\mathrm{Tor}_n^B(A, K) = 0 \text{ for } n > 0.$$

*Then*

$$Hk^n(A, B, \varepsilon) = \mathrm{Ext}_A^n(A \otimes_B K, A \otimes_B K) = \mathrm{Ext}_B^n(K, A \otimes_B K).$$

*In particular*

$$Hk^n(A, B, \varepsilon) = 0, \;\; n < 0,$$

*and*

$$Hk^0(A, B, \varepsilon) = \mathrm{Hom}_A(A \otimes_B K, A \otimes_B K)$$

*as an associative algebra.*

*Proof.* Equip the complex $Y = \mathrm{End}_A(A \otimes T(I(B)) \otimes K)$, which we used in Proposition 2.2.1 for the computation of $Hk^*(A, B, \varepsilon)$, with the first filtration as follows:

$$F^k Y = \sum_{n=-\infty}^{\infty} \prod_{p+q=n, p \geq k} Y^{p,q}.$$

The associated graded complex with respect to this filtration is the double direct sum

$$\mathrm{Gr} Y = \sum_{p,q=-\infty}^{\infty} Y^{p,q}.$$

One can show that the filtration is regular and the second term of the corresponding spectral sequence is

$$E_2^{p,q} = H_{d'}^p(H_{d''}^q(\mathrm{Gr} Y)), \tag{2.4.1}$$

where $H_{d'}^*$ and $H_{d''}^*$ denote the homologies of the complex with respect to the partial differentials (2.1.2).

Now observe that at the same time the complex $A \otimes T(I(B)) \otimes K$ is a complex for the calculation of $\mathrm{Tor}_n^B(A, K)$ because $A \otimes T(I(B)) \otimes B$ is a free resolution of $A$ as a right $B$–module. It is also free as a left $A$–module. Therefore the functor $\mathrm{Hom}_A(A \otimes T(I(B)) \otimes K, \cdot)$ is exact. By assumption $H^*(A \otimes T(I(B)) \otimes K) = \mathrm{Tor}_0^B(A, K) = A \otimes_B K$. Using the last two observations we can calculate the cohomology of the complex $\mathrm{Gr}Y$ with respect to the differential $d''$ :

$$\begin{aligned} &H_{d''}^*(\mathrm{Gr}Y) = H_{d''}^*(\mathrm{Hom}_A(A \otimes T(I(B)) \otimes K, A \otimes T(I(B)) \otimes K)) = \\ &\mathrm{Hom}_A(A \otimes T(I(B)) \otimes K, A \otimes_B K). \end{aligned} \tag{2.4.2}$$

Here $\mathrm{Hom}_A$ should be thought of as a direct sum of the double graded components. Now (2.4.2) provides that the spectral sequence (2.4.1) degenerates at the second term. Moreover,

$$E_2^{p,*} = H^p_{d'}(H^0_{d''}(\mathrm{Gr}Y)) = H^p_{d'}(\mathrm{Hom}_A(A \otimes T(I(B)) \otimes K, A \otimes_B K)).$$

But the complex $A \otimes T(I(B)) \otimes K$ may be regarded as a free resolution of the left $A$–module $A \otimes_B K$. Therefore

$$E_2^{p,*} = \mathrm{Ext}^p_A(A \otimes_B K, A \otimes_B K).$$

Finally by Theorem 5.12, [4] we have:

$$Hk^n(A, B, \varepsilon) = H^n(Y) = E_2^{n,0} = \mathrm{Ext}^n_A(A \otimes_B K, A \otimes_B K).$$

Since $\mathrm{Tor}^B_n(A, K) = 0$ for $n > 0$ we can apply the Shapiro lemma (see Proposition 4.1.3 in [4]) to simplify the last expression:

$$\mathrm{Ext}^n_A(A \otimes_B K, A \otimes_B K) = \mathrm{Ext}^n_B(K, A \otimes_B K).$$

Clearly, $Hk^0(A, B, \varepsilon) = \mathrm{Hom}_A(A \otimes_B K, A \otimes_B K)$ as an associative algebra. This completes the proof.

**Remark 2.4.1** *In particular the conditions of the theorem are satisfied if $A$ is projective as a right $B$–module. For instance suppose that there exists a subspace $N \subset A$ such that multiplication in $A$ provides an isomorphism of vector spaces $A \cong N \otimes B$. Then $A$ is a free right $B$–module.*

## 2.5 Comparison with the BRST complex

Let $\mathfrak{g}$ be a Lie algebra over a field $K$. For simplicity we suppose that $\mathfrak{g}$ is finite–dimensional. However the arguments presented below remain true, with some technical modifications, for an arbitrary Lie algebra. We shall apply the construction of Section 2.2 in the following situation.

Let $B = U(\mathfrak{g})$ and let $A$ be an associative algebra over $K$ containing $B$ as a subalgebra. Note that $U(\mathfrak{g})$ is naturally augmented. Consider the $U(\mathfrak{g})$–free resolution of the left $U(\mathfrak{g})$–module $K$ as follows:

$$X = U(\mathfrak{g}) \otimes \Lambda(\mathfrak{g}),$$
$$d(u \otimes x_1 \wedge \ldots \wedge x_n) = \sum_{i=1}^n (-1)^{i+1} u x_i \otimes x_1 \wedge \ldots \wedge \widehat{x_i} \wedge \ldots \wedge x_n +$$
$$\sum_{1 \le i < j \le n} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \widehat{x_i} \wedge \ldots \wedge \widehat{x_j} \wedge \ldots \wedge x_n,$$

where the symbol $\widehat{x_i}$ indicates that $x_i$ is to be omitted.

Introduce operators of exterior and inner multiplication on $\Lambda(\mathfrak{g})$ as follows. For every $x \in \mathfrak{g}$ and $x^* \in \mathfrak{g}^*$ we define

$$\overline{x}x_1 \wedge \ldots \wedge x_n = x \wedge x_1 \wedge \ldots \wedge x_n,$$

$$\overline{x^*}x_1 \wedge \ldots \wedge x_n = \textstyle\sum_{i=1}^n (-1)^{i+1} x^*(x_i) x_1 \wedge \ldots \wedge \widehat{x_i} \wedge \ldots \wedge x_n.$$

Equip the linear space $\mathfrak{g}+\mathfrak{g}^*$ with a scalar product given by the canonical paring between $\mathfrak{g}$ and $\mathfrak{g}^*$. Using this scalar product we can construct the Clifford algebra $C(\mathfrak{g}+\mathfrak{g}^*)$. The operators $\overline{x}, \overline{y^*}, x \in \mathfrak{g}, y^* \in \mathfrak{g}^*$ satisfy the defining relations of this algebra,

$$\overline{x}\,\overline{y^*} + \overline{y^*}\,\overline{x} = y^*(x).$$

Therefore the algebra $C(\mathfrak{g}+\mathfrak{g}^*)$ naturally acts in the space $\Lambda(\mathfrak{g})$. Moreover, it is well–known that $\mathrm{End}_K(\Lambda(\mathfrak{g})) = C(\mathfrak{g}+\mathfrak{g}^*)$.

Now the differential of the complex $A \otimes_{U(\mathfrak{g})} X = A \otimes \Lambda(\mathfrak{g})$ may be explicitly described using the operators of exterior and inner multiplications,

$$d = \sum_i e_i \otimes \overline{e_i^*} - \sum_{i,j} 1 \otimes \overline{[e_i, e_j]}\,\overline{e_i^* e_j^*}. \tag{2.5.1}$$

Here $e_i$ is a linear basis of $\mathfrak{g}$, $e_i^*$ is the dual basis, $e_i \otimes 1$ is regarded as the operator of right multiplication in $A$, $e_i \otimes 1 \cdot u \otimes 1 = ue_i \otimes 1$.

Now consider the complex $\mathrm{End}_A(A \otimes_{U(\mathfrak{g})} X) = \mathrm{End}_A(A \otimes \Lambda(\mathfrak{g}))$ for the computation of the algebra $Hk(A, B, \varepsilon)$. Observe that

$$\mathrm{End}_A(A \otimes \Lambda(\mathfrak{g})) = A^{opp} \otimes \mathrm{End}_K(\Lambda(\mathfrak{g})) = A^{opp} \otimes C(\mathfrak{g}+\mathfrak{g}^*).$$

Under this identification $A^{opp}$ acts on $A \otimes \Lambda(\mathfrak{g})$ by multiplication in $A$ on the right and the Clifford algebra acts by the exterior and inner multiplication in $\Lambda(\mathfrak{g})$. This allows to consider the differential (2.5.1) as an element of the complex $A^{opp} \otimes C(\mathfrak{g}+\mathfrak{g}^*)$.

It is easy to see that the canonical $\mathbb{Z}$–grading of the complex $A^{opp} \otimes C(\mathfrak{g}+\mathfrak{g}^*)$ coincides mod 2 with the $\mathbb{Z}_2$–grading inherited from the Clifford algebra. Therefore according to (2.1.1) the differential $\mathbf{d}$ is given by the supercommutator in $A^{opp} \otimes C(\mathfrak{g}+\mathfrak{g}^*)$ by element (2.5.1).

Now recall that the complex $A^{opp} \otimes C(\mathfrak{g}+\mathfrak{g}^*)$ with the differential given by the supercommutator by the element (2.5.1) is the quantum BRST complex proposed in [19]. This establishes

**Proposition 2.5.1** *The complex* $(\mathrm{End}_A(A \otimes_{U(\mathfrak{g})} X), \mathbf{d})$ *is isomorphic to the BRST one* $A^{opp} \otimes C(\mathfrak{g} + \mathfrak{g}^*)$ *with the differential being the supercommutator by the element (2.5.1).*

## 2.6 Whittaker model as a Hecke algebra

In this section we use the notation introduced in Section 1.1. Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{n}_+ \subset \mathfrak{g}$ the maximal nilpotent subalgebra, $\chi : \mathfrak{n}_+ \to \mathbb{C}$ a character. Let $W(\mathfrak{b}_-)$ be the Whittaker model of the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$.

**Proposition 2.6.1** *Suppose that the character* $\chi$ *is non–singular. Then* $W(\mathfrak{b}_-)$ *is isomorphic to* $Hk^0(U(\mathfrak{g}), U(\mathfrak{n}_+), \chi)^{opp}$ *as an associative algebra.*

*Proof.* First observe that since $\mathfrak{g} = \mathfrak{b}_- \oplus \mathfrak{n}_+$ we have a linear isomorphism $U(\mathfrak{g}) = U(\mathfrak{b}_-) \otimes U(\mathfrak{n}_+)$. Therefore from Remark 2.4.1 and Theorem 2.4.1 it follows that $Hk^0(U(\mathfrak{g}), U(\mathfrak{n}_+), \chi) = \mathrm{Hom}_{U(\mathfrak{g})}(Y_\chi, Y_\chi)$, where $Y_\chi = U(\mathfrak{g}) \otimes_{U(\mathfrak{n}_+)} \mathbb{C}_\chi$.

Now observe that the map

$$\mathrm{Hom}_{U(\mathfrak{g})}(Y_\chi, Y_\chi) \to \mathrm{Hom}_{U(\mathfrak{n}_+)}(\mathbb{C}_\chi, Y_\chi); \quad \tilde{v} \mapsto \hat{v},$$

where $\hat{v}$ is given by $\hat{v}(z) = \tilde{v}(1 \otimes z)$ for every $z \in \mathbb{C}_\chi$, is a linear isomorphism.

Note also that by Remark A and Theorem B there exists a linear isomorphism

$$\mathrm{Hom}_{U(\mathfrak{n}_+)}(\mathbb{C}_\chi, Y_\chi) \to W(\mathfrak{b}_-); \quad \hat{v} \mapsto v, \text{ where } v \otimes 1 = \hat{v}(1).$$

Therefore we have a linear isomorphism

$$\mathrm{Hom}_{U(\mathfrak{g})}(Y_\chi, Y_\chi) \to W(\mathfrak{b}_-); \quad \tilde{v} \mapsto v. \tag{2.6.1}$$

We have to prove that (2.6.1) is an antihomomorphism.

Let $\tilde{v}, \tilde{w} \in \mathrm{Hom}_{U(\mathfrak{g})}(Y_\chi, Y_\chi)$ be two elements such that $\tilde{v}(1 \otimes 1) = v \otimes 1$, $\tilde{w}(1 \otimes 1) = w \otimes 1$. Then $\tilde{v}(\tilde{w}(1 \otimes 1)) = \tilde{v}(w \otimes 1)$. Since $\tilde{v}$ is an $U(\mathfrak{g})$ endomorphism of $Y_\chi$ we have $\tilde{v}(\tilde{w}(1 \otimes 1)) = w\tilde{v}(1 \otimes 1) = wv \otimes 1$ This completes the proof.

# Chapter 3

# Quantum deformation of the Whittaker model

Let $\mathfrak{g}$ be a complex simple Lie algebra, $U_h(\mathfrak{g})$ the standard quantum group associated with $\mathfrak{g}$. In this section we construct a generalization of the Whittaker model $W(\mathfrak{b}_-)$ for $U_h(\mathfrak{g})$.

Let $U_h(\mathfrak{n}_+)$ be the subalgebra of $U_h(\mathfrak{g})$ corresponding to the nilpotent Lie subalgebra $\mathfrak{n}_+$. $U_h(\mathfrak{n}_+)$ is generated by simple positive root generators of $U_h(\mathfrak{g})$ subject to the quantum Serre relations. It is easy to show that $U_h(\mathfrak{n}_+)$ has no non–singular characters (taking nonvanishing values on all simple root generators). Our first main result is a family of new realizations of the quantum group $U_h(\mathfrak{g})$, one for each Coxeter element in the corresponding Weyl group (see also [27]). The counterparts of $U(\mathfrak{n}_+)$, which naturally arise in these new realizations of $U_h(\mathfrak{g})$, do have non–singular characters.

Using these new realizations we can immediately formulate a quantum group version of Definition A. We also prove counterparts of Theorems A and B for $U_h(\mathfrak{g})$.

Finally we define quantum group generalizations of the Toda Hamiltonians. In the spirit of quantum harmonic analysis these new Hamiltonians are difference operators. An alternative definition of these Hamiltonians has been recently given in [10].

## 3.1 Quantum groups

In this section we recall some basic facts about quantum groups. We follow the notation of [6].

Let $h$ be an indeterminate, $\mathbb{C}[[h]]$ the ring of formal power series in $h$. We shall consider $\mathbb{C}[[h]]$–modules equipped with the so–called $h$–adic topology. For every such module $V$ this topology is characterized by requiring that $\{h^n V \mid n \geq 0\}$ is a base of the neighbourhoods of 0 in $V$, and that translations in $V$ are continuous. It is easy to see that, for modules equipped with this topology, every $\mathbb{C}[[h]]$–module map is automatically continuous.

A topological Hopf algebra over $\mathbb{C}[[h]]$ is a complete $\mathbb{C}[[h]]$–module $A$ equipped with a structure of $\mathbb{C}[[h]]$–Hopf algebra (see [6], Definition 4.3.1), the algebraic tensor products entering the axioms of the Hopf algebra are replaced by their completions in the $h$–adic topology. We denote by $\mu, \imath, \Delta, \varepsilon, S$ the multiplication, the unit, the comultiplication, the counit and the antipode of $A$, respectively.

The standard quantum group $U_h(\mathfrak{g})$ associated to a complex finite–dimensional simple Lie algebra $\mathfrak{g}$ is the algebra over $\mathbb{C}[[h]]$ topologically generated by elements $H_i,\ X_i^+,\ X_i^-,\ i = 1, \ldots, l$, and with the following defining relations:

$$[H_i, H_j] = 0, \quad [H_i, X_j^\pm] = \pm a_{ij} X_j^\pm,$$

$$X_i^+ X_j^- - X_j^- X_i^+ = \delta_{i,j} \tfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{3.1.1}$$

$$\text{where } K_i = e^{d_i h H_i}, \quad e^h = q, \quad q_i = q^{d_i} = e^{d_i h},$$

and the quantum Serre relations:

$$\textstyle\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \begin{array}{c} 1 - a_{ij} \\ r \end{array} \right]_{q_i} (X_i^\pm)^{1-a_{ij}-r} X_j^\pm (X_i^\pm)^r = 0, \ i \neq j,$$

where

$$\left[ \begin{array}{c} m \\ n \end{array} \right]_q = \tfrac{[m]_q!}{[n]_q![n-m]_q!}, \ [n]_q! = [n]_q \ldots [1]_q, \ [n]_q = \tfrac{q^n - q^{-n}}{q - q^{-1}}.$$

$U_h(\mathfrak{g})$ is a topological Hopf algebra over $\mathbb{C}[[h]]$ with comultiplication defined by

$$\Delta_h(H_i) = H_i \otimes 1 + 1 \otimes H_i,$$

$$\Delta_h(X_i^+) = X_i^+ \otimes K_i + 1 \otimes X_i^+,$$

$$\Delta_h(X_i^-) = X_i^- \otimes 1 + K_i^{-1} \otimes X_i^-,$$

antipode defined by

$$S_h(H_i) = -H_i, \quad S_h(X_i^+) = -X_i^+ K_i^{-1}, \quad S_h(X_i^-) = -K_i X_i^-,$$

and counit defined by

$$\varepsilon_h(H_i) = \varepsilon_h(X_i^\pm) = 0.$$

We shall also use the weight–type generators defined by

$$Y_i = \sum_{j=1}^{l} d_i (a^{-1})_{ij} H_j,$$

and the elements $L_i = e^{hY_i}$. They commute with the root vectors $X_i^\pm$ as follows:

$$L_i X_j^\pm L_i^{-1} = q_i^{\pm\delta_{ij}} X_j^\pm. \tag{3.1.2}$$

The Hopf algebra $U_h(\mathfrak{g})$ is a quantization of the standard bialgebra structure on $\mathfrak{g}$, i.e. $U_h(\mathfrak{g})/hU_h(\mathfrak{g}) = U(\mathfrak{g})$, $\Delta_h = \Delta \pmod h$, where $\Delta$ is the standard comultiplication on $U(\mathfrak{g})$, and

$$\frac{\Delta_h - \Delta_h^{opp}}{h} \pmod h = \delta,$$

where $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is the standard cocycle on $\mathfrak{g}$. Recall that

$$\delta(x) = (\mathrm{ad}_x \otimes 1 + 1 \otimes \mathrm{ad}_x) 2r_+, \quad r_+ \in \mathfrak{g} \otimes \mathfrak{g},$$

$$r_+ = \frac{1}{2} \sum_{i=1}^{l} Y_i \otimes X_i + \sum_{\beta \in \Delta_+} (X_\beta, X_{-\beta})^{-1} X_\beta \otimes X_{-\beta}. \tag{3.1.3}$$

Here $X_{\pm\beta} \in \mathfrak{g}_{\pm\beta}$ are root vectors of $\mathfrak{g}$. The element $r_+ \in \mathfrak{g} \otimes \mathfrak{g}$ is called a classical r–matrix.

The following proposition describes the algebraic structure of $U_h(\mathfrak{g})$.

**Proposition 3.1.1 ([6], Proposition 6.5.5)** *Let $\mathfrak{g}$ be a finite–dimensional complex simple Lie algebra, let $U_h(\mathfrak{h})$ be the subalgebra of $U_h(\mathfrak{g})$ topologically generated by the $H_i, i = 1, \ldots l$. Then, there is an isomorphism of algebras $\varphi : U_h(\mathfrak{g}) \to U(\mathfrak{g})[[h]]$ over $\mathbb{C}[[h]]$ such that $\varphi = id \pmod h$ and $\varphi|_{U_h(\mathfrak{h})} = id$.*

**Proposition 3.1.2 ([6], Proposition 6.5.7)** *If $\mathfrak{g}$ is a finite–dimensional complex simple Lie algebra, the center $Z_h(\mathfrak{g})$ of $U_h(\mathfrak{g})$ is canonically isomorphic to $Z(\mathfrak{g})[[h]]$, where $Z(\mathfrak{g})$ is the center of $U(\mathfrak{g})$.*

**Corollary 3.1.3 ([6], Corollary 6.5.6)** *If $\mathfrak{g}$ be a finite–dimensional complex simple Lie algebra, then the assignment $V \mapsto V[[h]]$ is a one–to–one correspondence between the finite–dimensional irreducible representations of $\mathfrak{g}$ and indecomposable representations of $U_h(\mathfrak{g})$ which are free and of finite rank as $\mathbb{C}[[h]]$–modules. Furthermore for every such $V$ the action of the generators $H_i \in U_h(\mathfrak{g}),\quad i = 1, \ldots l$ on $V[[h]]$ coincides with the action of the root generators $H_i \in \mathfrak{h},\quad i = 1, \ldots l$.*

The representations of $U_h(\mathfrak{g})$ defined in the previous corollary are called finite–dimensional representations of $U_h(\mathfrak{g})$. For every finite–dimensional representation $\pi_V : \mathfrak{g} \to \mathrm{End}V$ we denote the corresponding representation of $U_h(\mathfrak{g})$ in the space $V[[h]]$ by the same letter.

$U_h(\mathfrak{g})$ is a quasitriangular Hopf algebra, i.e. there exists an invertible element $\mathcal{R} \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$, called a universal R–matrix, such that

$$\Delta_h^{opp}(a) = \mathcal{R}\Delta_h(a)\mathcal{R}^{-1} \text{ for all } a \in U_h(\mathfrak{g}), \tag{3.1.4}$$

where $\Delta^{opp} = \sigma\Delta$, $\sigma$ is the permutation in $U_h(\mathfrak{g})^{\otimes 2}$, $\sigma(x \otimes y) = y \otimes x$, and

$$\begin{aligned} (\Delta_h \otimes id)\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{23}, \\ (id \otimes \Delta_h)\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{12}, \end{aligned} \tag{3.1.5}$$

where $\mathcal{R}_{12} = \mathcal{R} \otimes 1,\ \mathcal{R}_{23} = 1 \otimes \mathcal{R},\ \mathcal{R}_{13} = (\sigma \otimes id)\mathcal{R}_{23}$.

From (3.1.4) and (3.1.5) it follows that $\mathcal{R}$ satisfies the quantum Yang–Baxter equation:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{3.1.6}$$

For every quasitriangular Hopf algebra we also have (see Proposition 4.2.7 in [6]):

$$(S \otimes id)\mathcal{R} = \mathcal{R}^{-1},$$

and

$$(S \otimes S)\mathcal{R} = \mathcal{R}. \tag{3.1.7}$$

We shall explicitly describe the element $\mathcal{R}$. First following [15] we recall the construction of root vectors of $U_h(\mathfrak{g})$. We shall use the so–called normal ordering in the root system $\Delta_+ = \{\beta_1, \ldots, \beta_N\}$ (see [32]).

**Definition 3.1.1** *An ordering of the root system $\Delta_+$ is called normal if all simple roots are written in an arbitrary order, and for any theree roots $\alpha,\ \beta,\ \gamma$ such that $\gamma = \alpha + \beta$ we have either $\alpha < \gamma < \beta$ or $\beta < \gamma < \alpha$.*

To construct root vectors we shall apply the following inductive algorithm. Let $\alpha, \beta, \gamma \in \Delta_+$ be positive roots such that $\gamma = \alpha + \beta,\ \alpha < \beta$ and $[\alpha, \beta]$ is the minimal segment including $\gamma$, i.e. the segment has no other roots $\alpha', \beta'$ such that $\gamma = \alpha' + \beta'$. Suppose that $X^{\pm}_{\alpha},\ X^{\pm}_{\beta}$ have already been constructed. Then we define

$$\begin{aligned} X^+_{\gamma} &= X^+_{\alpha} X^+_{\beta} - q^{(\alpha,\beta)} X^+_{\beta} X^+_{\alpha}, \\ X^-_{\gamma} &= X^-_{\beta} X^-_{\alpha} - q^{-(\alpha,\beta)} X^-_{\alpha} X^-_{\beta}. \end{aligned} \tag{3.1.8}$$

**Proposition 3.1.4** *For $\beta = \sum_{i=1}^{l} m_i \alpha_i,\ m_i \in \mathbb{N}$ $X^{\pm}_{\beta}$ is a polynomial in the noncommutative variables $X^{\pm}_i$ homogeneous in each $X^{\pm}_i$ of degree $m_i$.*

The root vectors $X_\beta$ satisfy the following relations:

$$[X^+_{\alpha}, X^-_{\alpha}] = a(\alpha) \frac{e^{h\alpha^\vee} - e^{-h\alpha^\vee}}{q - q^{-1}}.$$

where $a(\alpha) \in \mathbb{C}[[h]]$. They commute with elements of the subalgebra $U_h(\mathfrak{h})$ as follows:

$$[H_i, X^{\pm}_{\beta}] = \pm\beta(H_i) X^{\pm}_{\beta},\ i = 1, \ldots, l. \tag{3.1.9}$$

Note that by construction

$$X^+_{\beta} \pmod h = X_\beta \in \mathfrak{g}_\beta,$$

$$X^-_{\beta} \pmod h = X_{-\beta} \in \mathfrak{g}_{-\beta}$$

are root vectors of $\mathfrak{g}$. This implies that $a(\alpha) \pmod h = (X_\alpha, X_{-\alpha})$.

Let $U_h(\mathfrak{n}_+), U_h(\mathfrak{n}_-)$ be the $\mathbb{C}[[h]]$–subalgebras of $U_h(\mathfrak{g})$ topologically generated by the $X^+_i$ and by the $X^-_i$, respectively.

Now using the root vectors $X^{\pm}_{\beta}$ we can construct a topological basis of $U_h(\mathfrak{g})$. Define for $\mathbf{r} = (r_1, \ldots, r_N) \in \mathbb{N}^N$,

$$(X^+)^{\mathbf{r}} = (X^+_{\beta_1})^{r_1} \ldots (X^+_{\beta_N})^{r_N},$$

$$(X^-)^{\mathbf{r}} = (X^-_{\beta_1})^{r_1} \ldots (X^-_{\beta_N})^{r_N},$$

and for $\mathbf{s} = (s_1, \ldots s_l) \in \mathbb{N}^{\,l}$,

$$H^{\mathbf{s}} = H_1^{s_1} \ldots H_l^{s_l}.$$

**Proposition 3.1.5 ([15], Proposition 3.3)** *The elements* $(X^+)^{\mathbf{r}}$, $(X^-)^{\mathbf{t}}$ *and* $H^{\mathbf{s}}$, *for* $\mathbf{r},\ \mathbf{t} \in \mathbb{N}^N$, $\mathbf{s} \in \mathbb{N}^l$, *form topological bases of* $U_h(\mathfrak{n}_+), U_h(\mathfrak{n}_-)$ *and* $U_h(\mathfrak{h})$, *respectively, and the products* $(X^+)^{\mathbf{r}} H^{\mathbf{s}} (X^-)^{\mathbf{t}}$ *form a topological basis of* $U_h(\mathfrak{g})$. *In particular, multiplication defines an isomorphism of* $\mathbb{C}[[h]]$ *modules:*

$$U_h(\mathfrak{n}_-) \otimes U_h(\mathfrak{h}) \otimes U_h(\mathfrak{n}_+) \to U_h(\mathfrak{g}).$$

An explicit expression for $\mathcal{R}$ may be written by making use of the q-exponential

$$exp_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k)_q!},$$

where

$$(k)_q! = (1)_q \dots (k)_q, \quad (n)_q = \frac{q^n - 1}{q - 1}.$$

Now the element $\mathcal{R}$ may be written as (see Theorem 8.1 in [15]):

$$\mathcal{R} = exp\left[h \sum_{i=1}^{l} (Y_i \otimes H_i)\right] \prod_{\beta} exp_{q_\beta^{-1}}[(q - q^{-1}) a(\beta)^{-1} X_\beta^+ \otimes X_\beta^-], \quad (3.1.10)$$

where $q_\beta = q^{(\beta,\beta)}$; the product is over all the positive roots of $\mathfrak{g}$, and the order of the terms is such that the $\alpha$–term appears to the left of the $\beta$–term if $\alpha < \beta$ with respect to the normal ordering of $\Delta_+$.

**Remark 3.1.2** *The r–matrix* $r_+ = \frac{1}{2}h^{-1}(\mathcal{R} - 1 \otimes 1) \pmod h$, *which is the classical limit of* $\mathcal{R}$, *coincides with the classical r–matrix (3.1.3).*

## 3.2 Non–singular characters and quantum groups

In this section we construct quantum counterparts of the principal nilpotent Lie subalgebras of complex simple Lie algebras and of their non–singular characters. We mainly follow the exposition presented in [27].

First we would like to show that the algebra $U_h(\mathfrak{n}_+)$ spanned by $X_i^+, i = 1, \dots, l$ does not admit characters which take nonvanishing values on all generators $X_i^+$, except for the case of $U_h(sl(2))$ when the quantum Serre relations do not appear.

Suppose, $\chi_h$ is such a character, and $\chi_h(X_i^+) = c_i \in \mathbb{C}[[h]],\ c_i \neq 0,\ i = 1, \dots l$. By applying the character $\chi_h$ to the quantum Serre relations one

obtains a family of identities,

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \begin{array}{c} 1-a_{ij} \\ r \end{array} \right]_{q_i} = 0,\ i \neq j. \tag{3.2.1}$$

We claim that some of these relations fail for the quantized universal enveloping algebra $U_h(\mathfrak{g})$ of any simple Lie algebra $\mathfrak{g}$ , with the exception of $\mathfrak{g} = sl(2)$. In a more general setting, relations (3.2.1) are analysed in the following lemma.

**Lemma 3.2.1** *The only solutions of equation*

$$\sum_{k=0}^{m}(-1)^k \left[ \begin{array}{c} m \\ k \end{array} \right]_t t^{kc} = 0, \tag{3.2.2}$$

*where $t$ is an indeterminate, are of the form*

$$c = -m+1, -m+2, \ldots, m-2, m-1. \tag{3.2.3}$$

*Proof.* According to the q–binomial theorem [13],

$$\sum_{k=0}^{m}(-z)^k \left[ \begin{array}{c} m \\ k \end{array} \right]_t = \prod_{p=0}^{m-1}(1 - t^{m-1-2p}z). \tag{3.2.4}$$

Put $z = t^c$ in this relation. Then the l.h.s of (3.2.4) coincides with the l.h.s. of (3.2.2). Now (3.2.4) implies that $c = m-1-2p, p = 0, \ldots, m-1$ are the only solutions of (3.2.2).

Now we return to identities (3.2.1). Any Cartan matrix contains at least one off-diagonal element equal to $-1$. Then, $m = 1 - a_{ij} = 2$ and $c = \pm 1$, and Lemma 3.2.1 implies that some of identities (3.2.1) are false for any simple Lie algebra, except for $sl(2)$. Hence, subalgebras of $U_h(\mathfrak{g})$ generated by $X_i^+$ do not possess non–singular characters.

It is our goal to construct subalgebras of $U_h(\mathfrak{g})$ which resemble the subalgebra $U(\mathfrak{n}_+) \subset U(\mathfrak{g})$ and possess non–singular characters. Denote by $S_l$ the symmetric group of $l$ elements. To any element $\pi \in S_l$ we associate a Coxeter element $s_\pi$ by the formula $s_\pi = s_{\pi(1)} \ldots s_{\pi(l)}$. For each Coxeter element $s_\pi$ we define an associative algebra $U_h^{s_\pi}(\mathfrak{n}_+)$ generated by elements $e_i,\ i = 1, \ldots l$ subject to the relations :

$$\sum_{r=0}^{1-a_{ij}} (-1)^r q^{rc_{ij}^\pi} \left[ \begin{array}{c} 1-a_{ij} \\ r \end{array} \right]_{q_i} (e_i)^{1-a_{ij}-r} e_j (e_i)^r = 0,\ i \neq j, \tag{3.2.5}$$

where $c_{ij}^{\pi} = \left(\frac{1+s_\pi}{1-s_\pi}\alpha_i, \alpha_j\right)$ are matrix elements of the Caley transform of $s_\pi$ in the basis of simple roots.

**Proposition 3.2.2** *The map $\chi_h^{s_\pi} : U_h^{s_\pi}(\mathfrak{n}_+) \to \mathbb{C}[[h]]$ defined on generators by $\chi_h^{s_\pi}(e_i) = c_i,\ c_i \in \mathbb{C}[[h]],\ c_i \neq 0$ is a character of the algebra $U_h^{s_\pi}(\mathfrak{n}_+)$.*

To show that $\chi_h^{s_\pi}$ is a character of $U_h^{s_\pi}(\mathfrak{n}_+)$ it suffices to check that the defining relations (3.2.5) belong to the kernel of $\chi_h^{s_\pi}$, i.e.

$$\sum_{r=0}^{1-a_{ij}} (-1)^r q^{rc_{ij}^{\pi}} \left[ \begin{array}{c} 1-a_{ij} \\ r \end{array} \right]_{q_i} = 0,\ i \neq j. \tag{3.2.6}$$

As a preparation for the proof of Proposition 3.2.2 we study the matrix elements of the Caley transform of $s_\pi$ which enter the definition of $U_h^{s_\pi}(\mathfrak{n}_+)$.

**Lemma 3.2.3** *The matrix elements of $\frac{1+s_\pi}{1-s_\pi}$ are of the form :*

$$c_{ij}^{\pi} = \left(\frac{1+s_\pi}{1-s_\pi}\alpha_i, \alpha_j\right) = \varepsilon_{ij}^{\pi} b_{ij}, \tag{3.2.7}$$

*where*

$$\varepsilon_{ij}^{\pi} = \left\{ \begin{array}{ll} -1 & \pi^{-1}(i) < \pi^{-1}(j) \\ 0 & i = j \\ 1 & \pi^{-1}(i) > \pi^{-1}(j) \end{array} \right.$$

*Proof.* (compare [1], Ch. V, §6, Ex. 3). First we calculate the matrix of the Coxeter element $s_\pi$ with respect to the basis of simple roots. We obtain this matrix in the form of the Gauss decomposition of the operator $s_\pi$.

Let $z_{\pi(i)} = s_\pi \alpha_{\pi(i)}$. Recall that $s_i(\alpha_j) = \alpha_j - a_{ji}\alpha_i$. Using this definition the elements $z_{\pi(i)}$ may be represented as:

$$z_{\pi(i)} = y_{\pi(i)} - \sum_{k \geq i} a_{\pi(k)\pi(i)} y_{\pi(k)},$$

where

$$y_{\pi(i)} = s_{\pi(1)} \ldots s_{\pi(i-1)} \alpha_{\pi(i)}. \tag{3.2.8}$$

Using the matrix notation we can rewrite the last formula as follows:

$$z_{\pi(i)} = (I+V)_{\pi(k)\pi(i)} y_{\pi(k)},$$

$$\text{where } V_{\pi(k)\pi(i)} = \left\{ \begin{array}{ll} a_{\pi(k)\pi(i)} & k \geq i \\ 0 & k < i \end{array} \right. \tag{3.2.9}$$

To calculate the matrix of the operator $s_\pi$ with respect to the basis of simple roots we have to express the elements $y_{\pi(i)}$ via the simple roots. Applying the definition of simple reflections to (3.2.8) we can pull out the element $\alpha_{\pi(i)}$ to the right:

$$y_{\pi(i)} = \alpha_{\pi(i)} - \sum_{k<i} a_{\pi(k)\pi(i)} y_{\pi(k)}.$$

Therefore

$$\alpha_{\pi(i)} = (I+U)_{\pi(k)\pi(i)} y_{\pi(k)} \ , \text{ where } U_{\pi(k)\pi(i)} = \begin{cases} a_{\pi(k)\pi(i)} & k < i \\ 0 & k \geq i \end{cases}$$

Thus

$$y_{\pi(k)} = (I+U)^{-1}_{\pi(j)\pi(k)} \alpha_{\pi(j)}. \tag{3.2.10}$$

Summarizing (3.2.10) and (3.2.9) we obtain:

$$s_\pi \alpha_i = \left((I+U)^{-1}(I-V)\right)_{ki} \alpha_k. \tag{3.2.11}$$

This implies:

$$\frac{1+s_\pi}{1-s_\pi}\alpha_i = \left(\frac{2I+U-V}{U+V}\right)_{ki} \alpha_k. \tag{3.2.12}$$

Observe that $(U+V)_{ki} = a_{ki}$ and $(2I+U-V)_{ij} = -a_{ij}\varepsilon^\pi_{ij}$. Substituting these expressions into (3.2.12) we get :

$$\left(\frac{1+s_\pi}{1-s_\pi}\alpha_i, \alpha_j\right) = -(a^{-1})_{kp}\varepsilon^\pi_{pi} a_{pi} b_{jk} = \tag{3.2.13}$$

$$-d_j a_{jk} (a^{-1})_{kp} \varepsilon^\pi_{pi} a_{pi} = \varepsilon^\pi_{ij} b_{ij}. \tag{3.2.14}$$

This concludes the proof of the lemma.

*Proof of Proposition 3.2.2* Identities (3.2.6) follow from Lemma 3.2.1 for $t = q_i$, $m = 1 - a_{ij}$, $c = \varepsilon^\pi_{ij} a_{ij}$ since set of solutions (3.2.3) always contains $\pm(m-1)$.

Motivated by relations (3.2.5) we suggest new realizations of the quantum group $U_h(\mathfrak{g})$, one for each Coxeter element $s_\pi$. Let $U_h^{s_\pi}(\mathfrak{g})$ be the associative algebra over $\mathbb{C}[[h]]$ with generators $e_i, f_i, H_i$, $i = 1, \ldots l$ subject to the

relations:

$$
\begin{aligned}
&[H_i,H_j]=0, \quad [H_i,e_j]=a_{ij}e_j, \quad [H_i,f_j]=-a_{ij}f_j,\\
&e_if_j-q^{c^{\pi}_{ij}}f_je_i=\delta_{i,j}\tfrac{K_i-K_i^{-1}}{q_i-q_i^{-1}},\\
&K_i=e^{d_ihH_i},\\
&\textstyle\sum_{r=0}^{1-a_{ij}}(-1)^rq^{rc^{\pi}_{ij}}\left[\begin{array}{c}1-a_{ij}\\ r\end{array}\right]_{q_i}(e_i)^{1-a_{ij}-r}e_j(e_i)^r=0,\ i\neq j,\\
&\textstyle\sum_{r=0}^{1-a_{ij}}(-1)^rq^{rc^{\pi}_{ij}}\left[\begin{array}{c}1-a_{ij}\\ r\end{array}\right]_{q_i}(f_i)^{1-a_{ij}-r}f_j(f_i)^r=0,\ i\neq j.
\end{aligned}
\tag{3.2.15}
$$

It follows that the map $\tau_h^{\pi}:U_h^{s_\pi}(\mathfrak{n}_+)\rightarrow U_h^{s_\pi}(\mathfrak{g});\quad e_i\mapsto e_i$ is a *natural* embedding of $U_h^{s_\pi}(\mathfrak{n}_+)$ into $U_h^{s_\pi}(\mathfrak{g})$. From now on we identify $U_h^{s_\pi}(\mathfrak{n}_+)$ with the subalgebra in $U_h^{s_\pi}(\mathfrak{g})$ generated by $e_i, i=1,\ldots l$.

**Theorem 3.2.4** *For every solution $n_{ij}\in\mathbb{C},\ i,j=1,\ldots,l$ of equations*

$$d_jn_{ij}-d_in_{ji}=c^{\pi}_{ij} \tag{3.2.16}$$

*there exists an algebra isomorphism $\psi_{\{n\}}:U_h^{s_\pi}(\mathfrak{g})\rightarrow U_h(\mathfrak{g})$ defined by the formulas:*

$$
\begin{aligned}
&\psi_{\{n\}}(e_i)=X_i^+\textstyle\prod_{p=1}^lL_p^{n_{ip}},\\
&\psi_{\{n\}}(f_i)=\textstyle\prod_{p=1}^lL_p^{-n_{ip}}X_i^-,\\
&\psi_{\{n\}}(H_i)=H_i.
\end{aligned}
$$

*Proof* is provided by direct verification of defining relations (3.2.15). The most nontrivial part is to verify deformed quantum Serre relations (3.2.5). The defining relations of $U_h(\mathfrak{g})$ imply the following relations for $\psi_{\{n\}}(e_i)$,

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\left[\begin{array}{c}1-a_{ij}\\ k\end{array}\right]_{q_i}q^{k(d_jn_{ij}-d_in_{ji})}\psi_{\{n\}}(e_i)^{1-a_{ij}-k}\psi_{\{n\}}(e_j)\psi_{\{n\}}(e_i)^k=0,$$

for any $i\neq j$. Now using equation (3.2.16) we arrive to relations (3.2.5).

**Remark 3.2.3** *The general solution of equation (3.2.16) is given by*

$$n_{ji} = \frac{1}{2}(\varepsilon_{ij}^{\pi} a_{ij} + \frac{s_{ij}}{d_i}), \tag{3.2.17}$$

*where* $s_{ij} = s_{ji}$.

We call the algebra $U_h^{s_\pi}(\mathfrak{g})$ the Coxeter realization of the quantum group $U_h(\mathfrak{g})$ corresponding to the Coxeter element $s_\pi$.

**Remark 3.2.4** *Let* $n_{ij}$ *be a solution of the homogeneous system that corresponds to (3.2.16),*

$$d_i n_{ji} - d_j n_{ij} = 0.$$

*Then the map defined by*

$$\begin{aligned} &X_i^+ \mapsto X_i^+ \textstyle\prod_{p=1}^l L_p^{n_{ip}}, \\ &X_i^- \mapsto \textstyle\prod_{p=1}^l L_p^{-n_{ip}} X_i^-, \\ &H_i \mapsto H_i \end{aligned} \tag{3.2.18}$$

*is an automorphism of* $U_h(\mathfrak{g})$*. Therefore for given Coxeter element the isomorphism* $\psi_{\{n\}}$ *is defined uniquely up to automorphisms of* $U_h(\mathfrak{g})$*.*

Now we shall study the algebraic structure of $U_h^{s_\pi}(\mathfrak{g})$. Denote by $U_h^{s_\pi}(\mathfrak{n}_-)$ the subalgebra in $U_h^{s_\pi}(\mathfrak{g})$ generated by $f_i, i = 1, \ldots l$. From defining relations (3.2.15) it follows that the map $\overline{\chi}_h^{s_\pi} : U_h^{s_\pi}(\mathfrak{n}_-) \to \mathbb{C}[[h]]$ defined on generators by $\overline{\chi}_h^{s_\pi}(f_i) = c_i, c_i \in \mathbb{C}[[h]], c_i \neq 0$ is a character of the algebra $U_h^{s_\pi}(\mathfrak{n}_-)$.

Let $U_h^{s_\pi}(\mathfrak{h})$ be the subalgebra in $U_h^{s_\pi}(\mathfrak{g})$ generated by $H_i,\ i = 1, \ldots, l$. Define $U_h^{s_\pi}(\mathfrak{b}_\pm) = U_h^{s_\pi}(\mathfrak{n}_\pm)U_h^{s_\pi}(\mathfrak{h})$.

We shall construct a Poincaré–Birkhoff-Witt basis for $U_h^{s_\pi}(\mathfrak{g})$. It is convenient to introduce an operator $K \in \mathrm{End}\ \mathfrak{h}$ such that

$$KH_i = \sum_{j=1}^{l} \frac{n_{ij}}{d_i} Y_j. \tag{3.2.19}$$

In particular, we have

$$\frac{n_{ji}}{d_j} = (KH_j, H_i).$$

Equation (3.2.16) is equivalent to the following equation for the operator $K$:

$$K - K^* = \frac{1 + s_\pi}{1 - s_\pi}.$$

**Proposition 3.2.5** *(i)For any solution of equation (3.2.16) and any normal ordering of the root system $\Delta_+$ the elements $e_\beta = \psi_{\{n\}}^{-1}(X_\beta^+ e^{hK\beta^\vee})$ and $f_\beta = \psi_{\{n\}}^{-1}(e^{-hK\beta^\vee}X_\beta^-)$, $\beta \in \Delta_+$ lie in the subalgebras $U_h^{s_\pi}(\mathfrak{n}_+)$ and $U_h^{s_\pi}(\mathfrak{n}_-)$, respectively.*

*(ii)Moreover, the elements $e^{\mathbf{r}} = e_{\beta_1}^{r_1}\ldots e_{\beta_N}^{r_N}$, $f^{\mathbf{t}} = e_{\beta_1}^{t_1}\ldots e_{\beta_N}^{t_N}$ and $H^{\mathbf{s}} = H_1^{s_1}\ldots H_l^{s_l}$ for $\mathbf{r},\ \mathbf{t},\ \mathbf{s} \in \mathbb{N}^N$, form topological bases of $U_h^{s_\pi}(\mathfrak{n}_+)$, $U_h^{s_\pi}(\mathfrak{n}_-)$ and $U_h^{s_\pi}(\mathfrak{h})$, and the products $f^{\mathbf{t}}H^{\mathbf{s}}e^{\mathbf{r}}$ form a topological basis of $U_h^{s_\pi}(\mathfrak{g})$. In particular, multiplication defines an isomorphism of $\mathbb{C}[[h]]$ modules*

$$U_h^{s_\pi}(\mathfrak{n}_-)\otimes U_h^{s_\pi}(\mathfrak{h})\otimes U_h^{s_\pi}(\mathfrak{n}_+) \to U_h^{s_\pi}(\mathfrak{g}).$$

*Proof.* Let $\beta = \sum_{i=1}^l m_i\alpha_i \in \Delta_+$ be a positive root, $X_\beta^+ \in U_h(\mathfrak{g})$ the corresponding root vector. Then $\beta^\vee = \sum_{i=1}^l m_i d_i H_i$, and so $K\beta^\vee = \sum_{i,j=1}^l m_i n_{ij} Y_j$. Now the proof of the first statement follows immediately from Proposition 3.1.4, commutation relations (3.1.2) and the definition of the isomorphism $\psi_{\{n\}}$. The second assertion is a consequence of Proposition 3.1.5.

Now we would like to choose a normal ordering of the root system $\Delta_+$ in such a way that $\chi_h^{s_\pi}(e_\beta) = 0$ and $\overline{\chi}_h^{s_\pi}(f_\beta) = 0$ if $\beta$ is not a simple root.

**Proposition 3.2.6** *Choose a normal ordering of the root system $\Delta_+$ such that the simple roots are written in the following order: $\alpha_{\pi(1)},\ldots,\alpha_{\pi(l)}$. Then $\chi_h^{s_\pi}(e_\beta) = 0$ and $\overline{\chi}_h^{s_\pi}(f_\beta) = 0$ if $\beta$ is not a simple root.*

*Proof.* We shall consider the case of positive root generators. The proof for negative root generators is similar to that for the positive ones.

The root vectors $X_\beta^+$ are defined in terms of iterated q-commutators (see (3.1.8)). Therefore it suffices to verify that for $i < j$

$$\chi_h^{s_\pi}(e_{\alpha_{\pi(i)}+\alpha_{\pi(j)}}) =$$

$$\chi_h^{s_\pi}(\psi_{\{n\}}^{-1}((X_{\pi(i)}^+X_{\pi(j)}^+ - q^{(\alpha_{\pi(i)},\alpha_{\pi(j)})}X_{\pi(j)}^+X_{\pi(i)}^+)e^{hK(d_{\pi(i)}H_{\pi(i)}+d_{\pi(j)}H_{\pi(j)})})) = 0.$$

From (3.2.19) and commutation relations (3.1.2) we obtain that

$$\psi_{\{n\}}^{-1}((X_{\pi(i)}^+X_{\pi(j)}^+ - q^{(\alpha_{\pi(i)},\alpha_{\pi(j)})}X_{\pi(j)}^+X_{\pi(i)}^+)e^{hK(d_{\pi(i)}H_{\pi(i)}+d_{\pi(j)}H_{\pi(j)})}) =$$

$$q^{-d_{\pi(j)}n_{\pi(i)\pi(j)}}(e_{\pi(i)}e_{\pi(j)} - q^{b_{\pi(i)\pi(j)}+d_{\pi(j)}n_{\pi(i)\pi(j)}-d_{\pi(i)}n_{\pi(j)\pi(i)}}e_{\pi(j)}e_{\pi(i)}) \tag{3.2.20}$$

Now using equation (3.2.16) and Lemma 3.2.3 the combination $b_{\pi(i)\pi(j)} + d_{\pi(j)}n_{\pi(i)\pi(j)} - d_{\pi(i)}n_{\pi(j)\pi(i)}$ may be represented as $b_{\pi(i)\pi(j)} + \varepsilon_{\pi(i)\pi(j)}^{\pi}b_{\pi(i)\pi(j)}$.

But $\varepsilon^{\pi}_{\pi(i)\pi(j)} = -1$ for $i < j$ and therefore the r.h.s. of (3.2.20) takes the form

$$q^{-d_{\pi(j)}n_{\pi(i)\pi(j)}}[e_{\pi(i)}, e_{\pi(j)}].$$

Clearly,

$$\chi_h^{s_\pi}(e_{\alpha_{\pi(i)}+\alpha_{\pi(j)}}) = q^{-d_{\pi(j)}n_{\pi(i)\pi(j)}}\chi_h^{s_\pi}([e_{\pi(i)}, e_{\pi(j)}]) = 0.$$

## 3.3 Quantum deformation of the Whittaker model

In this section we define a quantum deformation of the Whittaker model $W(\mathfrak{b}_-)$. Our construction is similar the one described in Section 1.2, the quantum group $U_h^{s_\pi}(\mathfrak{g})$, the subalgebra $U_h^{s_\pi}(\mathfrak{n}_+)$ and characters $\chi_h^{s_\pi} : U_h^{s_\pi}(\mathfrak{n}_+) \rightarrow \mathbb{C}[[h]]$ serve as natural counterparts of the universal enveloping algebra $U(\mathfrak{g})$, of the subalgebra $U(\mathfrak{n}_+)$ and of non–singular characters $\chi : U(\mathfrak{n}_+) \rightarrow \mathbb{C}$.

Let $U_h^{s_\pi}(\mathfrak{n}_+)_{\chi_h^{s_\pi}}$ be the kernel of the character $\chi_h^{s_\pi} : U_h^{s_\pi}(\mathfrak{n}_+) \rightarrow \mathbb{C}[[h]]$ so that one has a direct sum

$$U_h^{s_\pi}(\mathfrak{n}_+) = \mathbb{C}[[h]] \oplus U_h^{s_\pi}(\mathfrak{n}_+)_{\chi_h^{s_\pi}}.$$

From Proposition 3.2.5 we have a linear isomorphism $U_h^{s_\pi}(\mathfrak{g}) = U_h^{s_\pi}(\mathfrak{b}_-)\otimes U_h^{s_\pi}(\mathfrak{n}_+)$ and hence the direct sum

$$U_h^{s_\pi}(\mathfrak{g}) = U_h^{s_\pi}(\mathfrak{b}_-) \oplus I_{\chi_h^{s_\pi}}, \tag{3.3.1}$$

where $I_{\chi_h^{s_\pi}} = U_h^{s_\pi}(\mathfrak{g})U_h^{s_\pi}(\mathfrak{n}_+)_{\chi_h^{s_\pi}}$ is the left–sided ideal generated by $U_h^{s_\pi}(\mathfrak{n}_+)_{\chi_h^{s_\pi}}$.

For any $u \in U_h^{s_\pi}(\mathfrak{g})$ let $u^{\chi_h^{s_\pi}} \in U_h^{s_\pi}(\mathfrak{b}_-)$ be its component in $U_h^{s_\pi}(\mathfrak{b}_-)$ relative to the decomposition (3.3.1). Denote by $\rho_{\chi_h^{s_\pi}}$ the linear map

$$\rho_{\chi_h^{s_\pi}} : U_h^{s_\pi}(\mathfrak{g}) \rightarrow U_h^{s_\pi}(\mathfrak{b}_-)$$

given by $\rho_{\chi_h^{s_\pi}}(u) = u^{\chi_h^{s_\pi}}$.

Denote by $Z_h^{s_\pi}(\mathfrak{g})$ the center of $U_h^{s_\pi}(\mathfrak{g})$. From Proposition 3.1.2 and Theorem 3.2.4 we obtain that $Z_h^{s_\pi}(\mathfrak{g}) \cong Z(\mathfrak{g})[[h]]$. In particular, $Z_h^{s_\pi}(\mathfrak{g})$ is freely generated as a commutative topological algebra over $\mathbb{C}[[h]]$ by $l$ elements $I_1, \ldots, I_l$.

Let $W_h(\mathfrak{b}_-) = \rho_{\chi_h^{s_\pi}}(Z_h^{s_\pi}(\mathfrak{g}))$.

**Theorem $\mathbf{A_h}$** *The map*

$$\rho_{\chi_h^{s_\pi}} : Z_h^{s_\pi}(\mathfrak{g}) \rightarrow W_h(\mathfrak{b}_-) \tag{3.3.2}$$

*is an isomorphism of algebras. In particular, $W_h(\mathfrak{b}_-)$ is freely generated as a commutative topological algebra over $\mathbb{C}[[h]]$ by $l$ elements $I_i^{\chi_h^{s_\pi}} = \rho_{\chi_h^{s_\pi}}(I_i), \ \ i = 1, \ldots, l$.*

*Proof* is similar to that of Theorem A in the classical case.

**Definition $\mathbf{A_h}$** *The algebra $W_h(\mathfrak{b}_-)$ is called the Whittaker model of $Z_h^{s_\pi}(\mathfrak{g})$.*

Next we equip $U_h^{s_\pi}(\mathfrak{b}_-)$ with a structure of a left $U_h^{s_\pi}(\mathfrak{n}_+)$ module in such a way that $W_h(\mathfrak{b}_-)$ is identified with the space of invariants with respect to this action. Following Lemma A in the classical case we define this action by

$$x \cdot v = [x, v]^{\chi_h^{s_\pi}}, \tag{3.3.3}$$

where $v \in U_h^{s_\pi}(\mathfrak{b}_-)$ and $x \in U_h^{s_\pi}(\mathfrak{n}_+)$.

Consider the space $U_h^{s_\pi}(\mathfrak{b}_-)^{U_h^{s_\pi}(\mathfrak{n}_+)}$ of $U_h^{s_\pi}(\mathfrak{n}_+)$ invariants in $U_h^{s_\pi}(\mathfrak{b}_-)$ with respect to this action. Clearly, $W_h(\mathfrak{b}_-) \subseteq U_h^{s_\pi}(\mathfrak{b}_-)^{U_h^{s_\pi}(\mathfrak{n}_+)}$.

**Theorem $\mathbf{B_h}$** *Suppose that $\chi_h^{s_\pi}(e_i) \neq 0 \pmod h$ for $i = 1, \ldots l$. Then the space of $U_h^{s_\pi}(\mathfrak{n}_+)$ invariants in $U_h^{s_\pi}(\mathfrak{b}_-)$ with respect to the action (3.3.3) is isomorphic to $W_h(\mathfrak{b}_-)$, i.e.*

$$U_h^{s_\pi}(\mathfrak{b}_-)^{U_h^{s_\pi}(\mathfrak{n}_+)} \cong W_h(\mathfrak{b}_-). \tag{3.3.4}$$

*Proof.* Let $p : U_h^{s_\pi}(\mathfrak{g}) \to U_h^{s_\pi}(\mathfrak{g})/hU_h^{s_\pi}(\mathfrak{g}) = U(\mathfrak{g})$ be the canonical projection. Note that $p(U_h^{s_\pi}(\mathfrak{n}_+)) = U(\mathfrak{n}_+)$, $p(U_h^{s_\pi}(\mathfrak{b}_-)) = U(\mathfrak{b}_-)$ and for every $x \in U_h^{s_\pi}(\mathfrak{n}_+)$ $\chi_h^{s_\pi}(x) \pmod h = \chi(p(x))$ for some non–singular character $\chi : U(\mathfrak{n}_+) \to \mathbb{C}$. Therefore $p(\rho_{\chi_h^{s_\pi}}(x)) = \rho_\chi(p(x))$ for every $x \in U_h^{s_\pi}(\mathfrak{g})$, and hence by Theorem $\mathrm{A}_q$ $p(W_h(\mathfrak{b}_-)) = W(\mathfrak{b}_-)$. Using Lemma A and the definition of action (3.3.3) we also obtain that $p(U_h^{s_\pi}(\mathfrak{b}_-)^{U_h^{s_\pi}(\mathfrak{n}_+)}) = U(\mathfrak{b}_-)^{N_+} = W(\mathfrak{b}_-)$.

Now let $I \in U_h^{s_\pi}(\mathfrak{b}_-)^{U_h^{s_\pi}(\mathfrak{n}_+)}$ be an invariant element. Then $p(I) \in W(\mathfrak{b}_-)$, and hence one can find an element $K_0 \in W_h(\mathfrak{b}_-)$ such that $I - K_0 = hI_1$, $I_1 \in U_h^{s_\pi}(\mathfrak{b}_-)^{U_h^{s_\pi}(\mathfrak{n}_+)}$. Applying the same procedure to $I_1$ one can find elements $K_1 \in W_h(\mathfrak{b}_-)$, $I_2 \in U_h^{s_\pi}(\mathfrak{b}_-)^{U_h^{s_\pi}(\mathfrak{n}_+)}$ such that $I_1 - K_1 = hI_2$, i.e. $I - K_0 - hK_1 = 0 \pmod{h^2}$. We can continue this process. Finally we obtain an infinite sequence of elements $K_i \in W_h(\mathfrak{b}_-)$ such that $I - \sum_{i=0}^p h^p K_p = 0 \pmod{h^{p+1}}$. Since the space $U_h^{s_\pi}(\mathfrak{b}_-)$ is complete in the $h$–adic topology the series $\sum_{i=0}^\infty h^p K_p \in W_h(\mathfrak{b}_-)$ converges to $I$. Therefore $I \in W_h(\mathfrak{b}_-)$. This completes the proof.

Similarly to Proposition 2.6.1 we have

**Proposition 3.3.1** *The algebra $W_h(\mathfrak{b}_-)$ is isomorphic to the zeroth graded component of the Hecke algebra of the triple $(U_h^{s_\pi}(\mathfrak{g}), U_h^{s_\pi}(\mathfrak{n}_+), \chi_h^{s_\pi})$ with the opposite multiplication,*

$$W_h(\mathfrak{b}_-) = Hk^0(U_h^{s_\pi}(\mathfrak{g}), U_h^{s_\pi}(\mathfrak{n}_+), \chi_h^{s_\pi})^{opp}.$$

## 3.4 Coxeter realizations of quantum groups and Drinfeld twist

In this section we show that the Coxeter realizations $U_h^{s_\pi}(\mathfrak{g})$ of the quantum group $U_h(\mathfrak{g})$ are connected with quantizations of some nonstandard bialgebra structures on $\mathfrak{g}$. At the quantum level changing bialgebra structure corresponds to the so–called Drinfeld twist. We shall consider a particular class of such twists described in the following proposition.

**Proposition 3.4.1 ([6], Proposition 4.2.13)** *Let $(A, \mu, \imath, \Delta, \varepsilon, S)$ be a Hopf algebra over a commutative ring. Let $\mathcal{F}$ be an invertible element of $A \otimes A$ such that*

$$\begin{gathered}\mathcal{F}_{12}(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{23}(id \otimes \Delta)(\mathcal{F}), \\ (\varepsilon \otimes id)(\mathcal{F}) = (id \otimes \varepsilon)(\mathcal{F}) = 1.\end{gathered} \tag{3.4.1}$$

*Then, $v = \mu(id \otimes S)(\mathcal{F})$ is an invertible element of $A$ with*

$$v^{-1} = \mu(S \otimes id)(\mathcal{F}^{-1}).$$

*Moreover , if we define $\Delta^{\mathcal{F}} : A \to A \otimes A$ and $S^{\mathcal{F}} : A \to A$ by*

$$\Delta^{\mathcal{F}}(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1}, \quad S^{\mathcal{F}}(a) = vS(a)v^{-1},$$

*then $(A, \mu, \imath, \Delta^{\mathcal{F}}, \varepsilon, S^{\mathcal{F}})$ is a Hopf algebra denoted by $A^{\mathcal{F}}$ and called the twist of $A$ by $\mathcal{F}$.*

**Corollary 3.4.2 ([6], Corollary 4.2.15)** *Suppose that $A$ and $\mathcal{F}$ as in Proposition 3.4.1, but assume in addition that $A$ is quasitriangular with universal R–matrix $\mathcal{R}$. Then $A^{\mathcal{F}}$ is quasitriangular with universal R–matrix*

$$\mathcal{R}^{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}, \tag{3.4.2}$$

*where $\mathcal{F}_{21} = \sigma\mathcal{F}$.*

Fix a Coxeter element $s_\pi \in W$, $s_\pi = s_{\pi(1)} \dots s_{\pi(l)}$. Consider the twist of the Hopf algebra $U_h(\mathfrak{g})$ by the element

$$\mathcal{F} = exp(-h \sum_{i,j=1}^{l} \frac{n_{ji}}{d_j} Y_i \otimes Y_j) \in U_h(\mathfrak{h}) \otimes U_h(\mathfrak{h}), \tag{3.4.3}$$

where $n_{ij}$ is a solution of the corresponding equation (3.2.16).

This element satisfies conditions (3.4.1), and so $U_h(\mathfrak{g})^{\mathcal{F}}$ is a quasitriangular Hopf algebra with the universal R–matrix $\mathcal{R}^{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}$, where $\mathcal{R}$ is given by (3.1.10). We shall explicitly calculate the element $\mathcal{R}^{\mathcal{F}}$. Substituting (3.1.10) and (3.4.3) into (3.4.2) and using (3.1.9) we obtain

$$\begin{array}{l}\mathcal{R}^{\mathcal{F}} = exp\left[h(\sum_{i=1}^{l}(Y_i \otimes H_i) + \sum_{i,j=1}^{l}(-\frac{n_{ij}}{d_i} + \frac{n_{ji}}{d_j})Y_i \otimes Y_j)\right] \times \\ \prod_\beta exp_{q_\beta^{-1}}[(q-q^{-1})a(\beta)^{-1}X_\beta^+ e^{hK\beta^\vee} \otimes e^{-hK^*\beta^\vee} X_\beta^-],\end{array}$$

where $K$ is defined by (3.2.19).

Equip $U_h^{s_\pi}(\mathfrak{g})$ with the comultiplication given by : $\Delta_{s_\pi}(x) = (\psi_{\{n\}}^{-1} \otimes \psi_{\{n\}}^{-1})\Delta_h^{\mathcal{F}}(\psi_{\{n\}}(x))$. Then $U_h^{s_\pi}(\mathfrak{g})$ becomes a quasitriangular Hopf algebra with the universal R–matrix $\mathcal{R}^{s_\pi} = \psi_{\{n\}}^{-1} \otimes \psi_{\{n\}}^{-1}\mathcal{R}^{\mathcal{F}}$. Using equation (3.2.16) and Lemma 3.2.3 this R–matrix may be written as follows

$$\begin{array}{l}\mathcal{R}^{s_\pi} = exp\left[h(\sum_{i=1}^{l}(Y_i \otimes H_i) + \sum_{i=1}^{l}\frac{1+s_\pi}{1-s_\pi}H_i \otimes Y_i)\right] \times \\ \prod_\beta exp_{q_\beta^{-1}}[(q-q^{-1})a(\beta)^{-1}e_\beta \otimes e^{h\frac{1+s_\pi}{1-s_\pi}\beta^\vee} f_\beta].\end{array} \tag{3.4.4}$$

The element $\mathcal{R}^{s_\pi}$ may be also represented in the form

$$\begin{array}{l}\mathcal{R}^{s_\pi} = exp\left[h(\sum_{i=1}^{l}(Y_i \otimes H_i)\right] \times \\ \prod_\beta exp_{q_\beta^{-1}}[(q-q^{-1})a(\beta)^{-1}e_\beta e^{-h\frac{1+s_\pi}{1-s_\pi}\beta^\vee} \otimes f_\beta] exp\left[h(\sum_{i=1}^{l}\frac{1+s_\pi}{1-s_\pi}H_i \otimes Y_i)\right].\end{array} \tag{3.4.5}$$

The comultiplication $\Delta_{s_\pi}$ is given on generators by

$$\Delta_{s_\pi}(H_i) = H_i \otimes 1 + 1 \otimes H_i,$$

$$\Delta_{s_\pi}(e_i) = e_i \otimes e^{hd_i\frac{2}{1-s_\pi}H_i} + 1 \otimes e_i,$$

$$\Delta_{s_\pi}(f_i) = f_i \otimes e^{-hd_i\frac{1+s_\pi}{1-s_\pi}H_i} + e^{-hd_iH_i} \otimes f_i.$$

Note that the Hopf algebra $U_h^{s_\pi}(\mathfrak{g})$ is a quantization of the bialgebra structure on $\mathfrak{g}$ defined by the cocycle

$$\delta(x) = (\mathrm{ad}_x \otimes 1 + 1 \otimes \mathrm{ad}_x) 2 r_+^{s_\pi}, \quad r_+^{s_\pi} \in \mathfrak{g} \otimes \mathfrak{g}, \tag{3.4.6}$$

where $r_+^{s_\pi} = r_+ + \frac{1}{2}\sum_{i=1}^{l} \frac{1+s_\pi}{1-s_\pi} H_i \otimes Y_i$, and $r_+$ is given by (3.1.3).

We shall also need the following property of the antipode $S^{s_\pi}$ of $U_h^{s_\pi}(\mathfrak{g})$.

**Proposition 3.4.3** *The square of the antipode $S^{s_\pi}$ is an inner automorphism of $U_h^{s_\pi}(\mathfrak{g})$ given by*

$$(S^{s_\pi})^2(x) = e^{2h\rho^\vee} x e^{-2h\rho^\vee},$$

*where $\rho^\vee = \sum_{i=1}^{l} Y_i$.*

*Proof.* First observe that by Proposition 3.4.1 the antipode of $U_h^{s_\pi}(\mathfrak{g})$ has the form: $S^{s_\pi}(x) = \psi_{\{n\}}^{-1}(v S_h(\psi_{\{n\}}(x)) v^{-1})$, where

$$v = exp(h \sum_{i,j=1}^{l} \frac{n_{ji}}{d_j} Y_i Y_j).$$

Therefore $(S^{s_\pi})^2(x) = \psi_{\{n\}}^{-1}(v S_h(v^{-1}) S_h^2(\psi_{\{n\}}(x)) S_h(v) v^{-1})$. Note that $S_h(v) = v$, and hence $(S^{s_\pi})^2(x) = \psi_{\{n\}}^{-1}(S_h^2(\psi_{\{n\}}(x)))$.

Finally observe that from explicit formulas for the antipode of $U_h(\mathfrak{g})$ it follows that $S_h^2(x) = e^{2h\rho^\vee} x e^{-2h\rho^\vee}$. This completes the proof.

In conclusion we note that using Corollary 3.1.3 and the isomorphism $\psi_{\{n\}}$ one can define finite–dimensional representations of $U_h^{s_\pi}(\mathfrak{g})$.

## 3.5 Quantum deformation of the Toda lattice

Recall that one of the main applications of the algebra $W(\mathfrak{b}_-)$ is the quantum Toda lattice [18]. Let $\overline{\chi} : \mathfrak{n}_- \to \mathbb{C}$ be a non–singular character of the opposite nilpotent subalgebra $\mathfrak{n}_-$. We denote the character of $N_-$ corresponding to $\overline{\chi}$ by the same letter. The algebra $U(\mathfrak{b}_-)$ naturally acts by differential operators in the space $C^\infty(\mathbb{C}_{\overline{\chi}} \otimes_{N_-} B_-)$. This space may be identified with $C^\infty(H)$. Let $D_1, \ldots, D_l$ be the differential operators on $C^\infty(H)$ which correspond to the elements $I_1^\chi, \ldots, I_l^\chi \in W(\mathfrak{b}_-)$. Denote by $\varphi$ the operator of multiplication in $C^\infty(H)$ by the function $\varphi(e^h) = e^{\rho(h)}$, where

$h \in \mathfrak{h}$. The operators $M_i = \varphi D_i \varphi^{-1}, i = 1, \dots l$ are called the quantum Toda Hamiltonians. Clearly, they commute with each other.

In particular if $I$ is the quadratic Casimir element then the corresponding operator $M$ is the well–known second–order differential operator:

$$M = \sum_{i=1}^{l} \partial_i^2 + \sum_{i=1}^{l} \chi(X_{\alpha_i})\overline{\chi}(X_{-\alpha_i})e^{-\alpha_i(h)} + (\rho, \rho),$$

where $\partial_i = \frac{\partial}{\partial y_i}$, and $y_i, \ i = 1, \dots l$ is an ortonormal basis of $\mathfrak{h}$.

Using the algebra $W_h(\mathfrak{b}_-)$ we shall construct quantum group analogues of the Toda Hamiltonians. A slightly different approach has been recently proposed in [10].

Denote by $A$ the space of linear functions on $\mathbb{C}[[h]]_{\overline{\chi}_h^{s_\pi}} \otimes_{U_h^{s_\pi}(\mathfrak{n}_-)} U_h^{s_\pi}(\mathfrak{b}_-)$, where $\mathbb{C}[[h]]_{\overline{\chi}_h^{s_\pi}}$ is the one–dimensional $U_h^{s_\pi}(\mathfrak{n}_-)$ module defined by $\overline{\chi}_h^{s_\pi}$. Note that $\mathbb{C}[[h]]_{\overline{\chi}_h^{s_\pi}} \otimes_{U_h^{s_\pi}(\mathfrak{n}_-)} U_h^{s_\pi}(\mathfrak{b}_-) \cong U_h^{s_\pi}(\mathfrak{h})$ as a linear space. Therefore $A = U_h^{s_\pi}(\mathfrak{h})^*$. The algebra $U_h^{s_\pi}(\mathfrak{b}_-)$ naturally acts on $\mathbb{C}[[h]]_{\overline{\chi}_h^{s_\pi}} \otimes_{U_h^{s_\pi}(\mathfrak{n}_-)} U_h^{s_\pi}(\mathfrak{b}_-)$ by multiplications from the right. This action induces an $U_h^{s_\pi}(\mathfrak{b}_-)$–action in the space $A$. We denote this action by $L$, $L : U_h^{s_\pi}(\mathfrak{b}_-) \rightarrow \mathrm{End}A$. Clearly, this action generates an action of the algebra $W_h(\mathfrak{b}_-)$ on $A$.

To construct deformed Toda Hamiltonians we shall use certain elements in $W_h(\mathfrak{b}_-)$. These elements may be described as follows. Let $\mu : U_h^{s_\pi}(\mathfrak{g}) \rightarrow \mathbb{C}[[h]]$ be a map such that $\mu(uv) = \mu(vu)$. By Proposition 3.4.3 $(S^{s_\pi})^2(x) = e^{2h\rho^\vee} x e^{-2h\rho^\vee}$. Hence from Remark 1 in [9] it follows that $(id \otimes \mu)(\mathcal{R}_{21}^{s_\pi}\mathcal{R}^{s_\pi}(1 \otimes e^{2h\rho^\vee}))$, where $\mathcal{R}_{21}^{s_\pi} = \sigma\mathcal{R}^{s_\pi}$, is a central element. In particular, for any finite–dimensional $\mathfrak{g}$–module $V$ the element

$$C_V = (id \otimes tr_V)(\mathcal{R}_{21}^{s_\pi}\mathcal{R}^{s_\pi}(1 \otimes e^{2h\rho^\vee})), \tag{3.5.1}$$

where $tr_V$ is the trace in $V[[h]]$, is central in $U_h^{s_\pi}(\mathfrak{g})$.

Using formulas (3.4.4) and (3.4.5) we can easily compute elements $\rho_{\chi_h^{s_\pi}}(C_V) \in W_h(\mathfrak{b}_-)$. For every finite–dimensional $\mathfrak{g}$–module $V$ we have

$$\rho_{\chi_h^{s_\pi}}(C_V) = (id \otimes tr_V)(e^{t_0} \prod_\beta exp_{q_\beta^{-1}}[(q - q^{-1})a(\beta)^{-1} f_\beta \otimes e_\beta e^{-h\frac{1+s_\pi}{1-s_\pi}\beta^\vee}] \times$$
$$e^{t_0} \prod_\beta exp_{q_\beta^{-1}}[(q - q^{-1})a(\beta)^{-1} \chi_h^{s_\pi}(e_\beta) \otimes e^{h\frac{1+s_\pi}{1-s_\pi}\beta^\vee} f_\beta](1 \otimes e^{2h\rho^\vee})), \tag{3.5.2}$$

where $t_0 = h\sum_{i=1}^{l}(Y_i \otimes H_i)$.

We denote by $W_h^{Rep}(\mathfrak{b}_-)$ the subalgebra in $W_h(\mathfrak{b}_-)$ generated by the elements $\rho_{\chi_h^{s_\pi}}(C_V)$, where $V$ runs through all finite–dimensional representations of $\mathfrak{g}$. Note that for every finite–dimensional $\mathfrak{g}$–module $V$ $\rho_{\chi_h^{s_\pi}}(C_V)$ is a polynomial in noncommutative elements $f_i,\ e^{hx},\ x \in \mathfrak{h}$.

Now we shall realize elements of $W_h^{Rep}(\mathfrak{b}_-)$ as difference operators. Let $H_h \in U_h^{s_\pi}(\mathfrak{h})$ be the subgroup generated by elements $e^{hx},\ x \in \mathfrak{h}$. A difference operator on $A$ is an operator $T$ of the form $T = \sum f_i T_{x_i}$ (a finite sum), where $f_i \in A$, and for every $y \in H_h$ $\ T_x f(y) = (ye^{hx}),\ x \in \mathfrak{h}$.

**Proposition 3.5.1 ([10], Proposition 3.2)** *For any $Y \in U_h^{s_\pi}(\mathfrak{b}_-)$, which is a polynomial in noncommutative elements $f_i,\ e^{hx},\ x \in \mathfrak{h}$, the operator $L(Y)$ is a difference operator on $A$. In particular, the operators $L(I),\ I \in W_h^{Rep}(\mathfrak{b}_-)$ are mutually commuting difference operators on $A$.*

*Proof.* It suffices to verify that $L(f_i)$ are difference operators on $H_h$. Indeed,

$$L(f_i)f(e^{hx}) = f(e^{hx}f_i) = e^{-h\alpha_i(x)}f(f_ie^{hx}) = \overline{\chi}_h^{s_\pi}(f_i)e^{-h\alpha_i(x)}f(e^{hx}).$$

This completes the proof.

Let $\jmath : H_h \to U_h^{s_\pi}(\mathfrak{h})$ be the canonical embedding. Denote $A_h = \jmath^*(A)$. Let $T$ be a difference operator on $A$. Then one can define a difference operator $\jmath^*(T)$ on the space $A_h$ by $\jmath^*(T)f(y) = T(\jmath(y))$.

Let $D_i^h = \jmath^*(L(\rho_{\chi_h^{s_\pi}}(C_{V_i})))$, where $V_i,\ i = 1, \ldots l$ are the fundamental representations of $\mathfrak{g}$. Denote by $\varphi_h$ the operator of multiplication in $A_h$ by the function $\varphi_h(e^{hx}) = e^{h\rho(x)}$, where $x \in \mathfrak{h}$. The operators $M_i^h = \varphi_h D_i^h \varphi_h^{-1}, i = 1, \ldots l$ are called the quantum deformed Toda Hamiltonians.

From now on we suppose that $\pi = id$ and that the ordering of positive roots $\Delta_+$ is fixed as in Proposition 3.2.6. We denote $s_{id} = s$. Now using formula (3.5.2) we outline computation of the operators $M_i^h$. This computation is simplified by the following lemma.

**Lemma 3.5.2 ([10], Lemma 5.2)** *Let $X = f_{\gamma_1} ... f_{\gamma_n}$. If the roots $\gamma_1, ..., \gamma_n$ are not all simple then $L(X) = 0$. Otherwise, if $\gamma_i = \alpha_{k_i}$, then*

$$\jmath^*(L(X))f(e^{hy}) = e^{-h(\sum \alpha_{k_i}, y)}f(e^{hy})\prod_i \overline{\chi}_h^s(f_{k_i})$$

*Proof* follows immediately from Proposition 3.2.6 and the arguments used in the proof of Proposition 3.5.1.

Using this lemma we obtain that if $\beta$ is not a simple root then the term in (3.5.2) containing root vector $f_\beta$ gives a trivial contribution to the operators

$L(\rho_{\chi_h^s}(C_{V_i}))$. Note also that by Proposition 3.2.6 $\chi_h^s(e_\beta) = 0$ if $\beta$ is not a simple root. Therefore from formula (3.5.2) we have

$$
\begin{aligned}
&L(\rho_{\chi_h^s}(C_{V_i})) = \\
&L(id \otimes tr_V)(e^{t_0} \prod_i exp_{q^{-2d_i}}[(q_i - q_i^{-1}) f_i \otimes e_i e^{-hd_i \frac{1+s}{1-s} H_i}] \times \\
&e^{t_0} \prod_i exp_{q^{-2d_i}}[(q_i - q_i^{-1}) \chi_h^s(e_i) \otimes e^{hd_i \frac{1+s}{1-s} H_i} f_i](1 \otimes e^{2h\rho^\vee})).
\end{aligned}
\tag{3.5.3}
$$

In particular, let $\mathfrak{g} = sl(n)$, $V_1 = V$ the fundamental representation of $sl(n)$. Then direct calculation gives

$$
M_1 f(e^{hy}) = \left( \sum_{j=1}^{n} T_j^2 - (q - q^{-1})^2 \sum_{i=1}^{n-1} \chi_h^s(e_i) \overline{\chi}_h^s(f_i) e^{-h(y,\alpha_i)} T_{i+1} T_i \right) f(e^{hy}),
$$

where $T_i = T_{\omega_i}$, $\omega_i$ are the weights of $V$. The last expression coincides with formula (5.7) obtained in [10].

# Chapter 4

# Poisson–Lie groups and Whittaker model

In this Chapter we introduce another quantum version of the Whittaker model. We consider quantizations of algebras of regular functions on algebraic Poisson–Lie groups. We define the Whittaker model of the center of these quantum algebras. The algebraic structure of this model is related to the structure of the set of regular elements in the corresponding algebraic group. This relation is parallel to the one established by Kostant for Lie algebras (see Section 1.3). Our main geometric result is an analog of Theorem C for algebraic groups. A generalization of this theorem for loop groups is contained in [26].

## 4.1 Poisson–Lie groups

Recall some notions concerned with Poisson–Lie groups (see [8], [22], [25], [6]). Let $G$ be a finite–dimensional Lie group equipped with a Poisson bracket, $\mathfrak{g}$ its Lie algebra. $G$ is called a Poisson–Lie group if the multiplication $G \times G \to G$ is a Poisson map. A Poisson bracket satisfying this axiom is degenerate and, in particular, is identically zero at the unit element of the group. Linearizing this bracket at the unit element defines the structure of a Lie algebra in the space $T_e^*G \simeq \mathfrak{g}^*$. The pair $(\mathfrak{g}, \mathfrak{g}^*)$ is called the tangent bialgebra of $G$.

Lie brackets in $\mathfrak{g}$ and $\mathfrak{g}^*$ satisfy the following compatibility condition:

*Let* $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ *be the dual of the commutator map* $[,]_* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$. *Then* $\delta$ *is a 1-cocycle on* $\mathfrak{g}$ *(with respect to the adjoint action of* $\mathfrak{g}$ *on* $\mathfrak{g} \wedge \mathfrak{g}$*).*

Let $c_{ij}^k, f_c^{ab}$ be the structure constants of $\mathfrak{g}, \mathfrak{g}^*$ with respect to the dual bases $\{e_i\}, \{e^i\}$ in $\mathfrak{g}, \mathfrak{g}^*$. The compatibility condition means that

$$c_{ab}^s f_s^{ik} \;-\; c_{as}^i f_b^{sk} \;+\; c_{as}^k f_b^{si} \;-\; c_{bs}^k f_a^{si} \;+\; c_{bs}^i f_a^{sk} \;=\; 0.$$

This condition is symmetric with respect to exchange of $c$ and $f$. Thus if $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, then $(\mathfrak{g}^*, \mathfrak{g})$ is also a Lie bialgebra.

The following proposition shows that the category of finite–dimensional Lie bialgebras is isomorphic to the category of finite–dimensional connected simply connected Poisson–Lie groups.

**Proposition 4.1.1 ([6], Theorem 1.3.2)** *If $G$ is a connected simply connected finite–dimensional Lie group, every bialgebra structure on $\mathfrak{g}$ is the tangent bialgebra of a unique Poisson structure on $G$ which makes $G$ into a Poisson–Lie group.*

Let $G$ be a finite–dimensional Poisson–Lie group, $(\mathfrak{g}, \mathfrak{g}^*)$ the tangent bialgebra of $G$. The connected simply connected finite–dimensional Poisson–Lie group corresponding to the Lie bialgebra $(\mathfrak{g}^*, \mathfrak{g})$ is called the dual Poisson–Lie group and denoted by $G^*$.

$(\mathfrak{g}, \mathfrak{g}^*)$ is called a *factorizable Lie bialgebra* if the following conditions are satisfied (see [22] , [8]):

1. *$\mathfrak{g}$ is equipped with a non–degenerate invariant scalar product* $(\cdot, \cdot)$.

   We shall always identify $\mathfrak{g}^*$ and $\mathfrak{g}$ by means of this scalar product.

2. *The dual Lie bracket on $\mathfrak{g}^* \simeq \mathfrak{g}$ is given by*

   $$[X, Y]_* = \frac{1}{2}\left([rX, Y] + [X, rY]\right), X, Y \in \mathfrak{g}, \tag{4.1.1}$$

   *where $r \in \mathrm{End}\ \mathfrak{g}$ is a skew symmetric linear operator (classical r-matrix).*

3. *$r$ satisfies the modified classical Yang-Baxter identity:*

   $$[rX, rY] - r\left([rX, Y] + [X, rY]\right) = -\left[X, Y\right],\ X, Y \in \mathfrak{g}. \tag{4.1.2}$$

Define operators $r_\pm \in \mathrm{End}\ \mathfrak{g}$ by

$$r_\pm = \frac{1}{2}\left(r \pm id\right).$$

We shall need some properties of the operators $r_\pm$. Denote by $\mathfrak{b}_\pm$ and $\mathfrak{n}_\mp$ the image and the kernel of the operator $r_\pm$:

$$\mathfrak{b}_\pm = Im\ r_\pm, \quad \mathfrak{n}_\mp = Ker\ r_\pm. \tag{4.1.3}$$

**Proposition 4.1.2 ([3], [23])** *Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a factorizable Lie bialgebra. Then*

*(i) $\mathfrak{b}_\pm \subset \mathfrak{g}$ is a Lie subalgebra, the subspace $\mathfrak{n}_\pm$ is a Lie ideal in $\mathfrak{b}_\pm$, $\mathfrak{b}_\pm^\perp = \mathfrak{n}_\pm$.*

*(ii) $\mathfrak{n}_\pm$ is an ideal in $\mathfrak{g}^*$.*

*(iii) $\mathfrak{b}_\pm$ is a Lie subalgebra in $\mathfrak{g}^*$. Moreover $\mathfrak{b}_\pm = \mathfrak{g}^*/\mathfrak{n}_\pm$.*

*(iv) $(\mathfrak{b}_\pm, \mathfrak{b}_\pm^*)$ is a subbialgebra of $(\mathfrak{g}, \mathfrak{g}^*)$ and $(\mathfrak{b}_\pm, \mathfrak{b}_\pm^*) \simeq (\mathfrak{b}_\pm, \mathfrak{b}_\mp)$. The canonical paring between $\mathfrak{b}_\mp$ and $\mathfrak{b}_\pm$is given by*

$$(X_\mp, Y_\pm)_\pm = (X_\mp, r_\pm^{-1} Y_\pm), \;\; X_\mp \in \mathfrak{b}_\mp; \;\; Y_\pm \in \mathfrak{b}_\pm. \tag{4.1.4}$$

The classical Yang–Baxter equation implies that $r_\pm$ , regarded as a mapping from $\mathfrak{g}^*$ into $\mathfrak{g}$, is a Lie algebra homomorphism. Moreover, $r_+^* = -r_-$, and $r_+ - r_- = id$.

Put $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ (direct sum of two copies). The mapping

$$\mathfrak{g}^* \to \mathfrak{d} \quad : X \mapsto (X_+, \; X_-), \quad X_\pm \;=\; r_\pm X \tag{4.1.5}$$

is a Lie algebra embedding. Thus we may identify $\mathfrak{g}^*$ with a Lie subalgebra in $\mathfrak{d}$.

Naturally, embedding (4.1.5) extends to an embedding

$$G^* \to G \times G, \quad L \mapsto (L_+, L_-).$$

We shall identify $G^*$ with the corresponding subgroup in $G \times G$.

## 4.2 Poisson reduction

In this section we recall basic facts on Poisson reduction (see [30], [24]). These facts will be used in the proof of the group counterpart of Theorem E (see Section 1.3).

Let $M, \; B, \; B'$ be Poisson manifolds. Two Poisson surjections

$$\begin{tikzcd} & M \arrow[dl, "\pi'"'] \arrow[dr, "\pi"] & \\ B' & & B \end{tikzcd}$$

form a dual pair if the pullback ${\pi'}^*C^\infty(B')$ is the centralizer of $\pi^*C^\infty(B)$ in the Poisson algebra $C^\infty(M)$. In that case the sets $B'_b = \pi'\left(\pi^{-1}(b)\right),\ b \in B$ are Poisson submanifolds in $B'$ (see [30]) called reduced Poisson manifolds.

Fix an element $b \in B$. Then the algebra of functions $C^\infty(B'_b)$ may be described as follows. Let $I_b$ be the ideal in $C^\infty(M)$ generated by elements $\pi^*(f),\ f \in C^\infty(B),\ f(b) = 0$. Denote $M_b = \pi^{-1}(b)$. Then the algebra $C^\infty(M_b)$ is simply the quotient of $C^\infty(M)$ by $I_b$. Denote by $P_b : C^\infty(M) \to C^\infty(M)/I_b = C^\infty(M_b)$ the canonical projection onto the quotient.

**Lemma 4.2.1** *Suppose that the map $f \mapsto f(b)$ is a character of the Poisson algebra $C^\infty(B)$. Then one can define an action of the Poisson algebra $C^\infty(B)$ on the space $C^\infty(M_b)$ by*

$$f \cdot \varphi = P_b(\{\pi^*(f), \tilde{\varphi}\}), \tag{4.2.1}$$

*where $f \in C^\infty(B)$, $\varphi \in C^\infty(M_b)$, $\tilde{\varphi} \in C^\infty(M)$ is a representative of $\varphi$ in $C^\infty(M)$ such that $P_b(\tilde{\varphi}) = \varphi$. Moreover, $C^\infty(B'_b)$ is the subspace of invariants in $C^\infty(M_b)$ with respect to this action.*

*Proof.* Let $\varphi \in C^\infty(M_b)$. Choose a representative $\tilde{\varphi} \in C^\infty(M)$ such that $P_b(\tilde{\varphi}) = \varphi$. Since the map $f \mapsto f(b)$ is a character of the Poisson algebra $C^\infty(B)$, Hamiltonian vector fields of functions $\pi^*(f),\ f \in C^\infty(B)$ are tangent to the surface $M_b$. Therefore using the the definition of the dual pair we obtain that $\varphi = \pi'^*(\psi)$ for some $\psi \in C^\infty(B'_b)$ if and only if $P_b(\{\pi^*(f), \tilde{\varphi}\}) = 0$ for every $f \in C^\infty(B)$. Note also that the r.h.s. of (4.2.1) only depends on $\varphi$ but not on the representative $\tilde{\varphi}$, and hence formula (4.2.1) defines an action of the Poisson algebra $C^\infty(B)$ on the space $C^\infty(M_b)$. Finally we obtain that $C^\infty(B'_b)$ is exactly the subspace of invariants in $C^\infty(M_b)$ with respect to this action.

**Definition 4.2.2** *The algebra $C^\infty(B'_b)$ is called a reduced Poisson algebra. We also denote it by $C^\infty(M_b)^{C^\infty(B)}$.*

**Remark 4.2.5** *Note that the description of the algebra $C^\infty(M_b)^{C^\infty(B)}$ obtained in Lemma 4.2.1 is independent of both the manifold $B'$ and the projection $\pi'$. Observe also that the reduced space $B'_b$ may be identified with a cross–section of the action of the Poisson algebra $C^\infty(B)$ on $M_b$ by Hamiltonian vector fields. In particular, $B'_b$ may be regarded as a submanifold in $M_b$.*

An important example of dual pairs is provided by Poisson group actions. Recall that a Poisson group action of a Poisson–Lie group $A$ on a Poisson manifold $M$ is a group action $A \times M \to M$ which is also a Poisson map (as usual, we suppose that $A \times M$ is equipped with the product Poisson structure). In [24] it is proved that if the space $M/A$ is a smooth manifold, there exists a unique Poisson structure on $M/A$ such that the canonical projection $M \to M/A$ is a Poisson map.

Let $\mathfrak{a}$ be the Lie algebra of $A$. Denote by $\langle \cdot, \cdot \rangle$ the canonical paring between $\mathfrak{a}^*$ and $\mathfrak{a}$. A map $\mu : M \to A^*$ is called a moment map for a right Poisson group action $A \times M \to M$ if ([20])

$$L_{\widehat{X}}\varphi = \langle \mu^*(\theta_{A^*}), X \rangle (\xi_\varphi), \tag{4.2.2}$$

where $\theta_{A^*}$ is the universal right–invariant Maurer–Cartan form on $A^*$, $X \in \mathfrak{a}$, $\widehat{X}$ is the corresponding vector field on $M$ and $\xi_\varphi$ is the Hamiltonian vector field of $\varphi \in C^\infty(M)$.

By Theorem 4.9, [20] one can always equip $A^*$ with a Poisson structure in such a way that $\mu$ becomes a Poisson mapping. Then from the definition of the moment map it follows that if $M/A$ is a smooth manifold then the canonical projection $M \to M/A$ and the moment map $\mu : M \to A^*$ form a dual pair (see [20] for details).

The main example of Poisson group actions is the so–called dressing action. The dressing action may be described as follows (see [20], [24]).

**Proposition 4.2.2** *Let $G$ be a connected simply connected Poisson–Lie group with factorizable tangent Lie bialgebra, $G^*$ the dual group. Then there exists a unique right Poisson group action*

$$G^* \times G \to G^*, \quad ((L_+, L_-), g) \mapsto g \circ (L_+, L_-),$$

*such that the identity mapping $\mu : G^* \to G^*$ is the moment map for this action.*

*Moreover, let $q : G^* \to G$ be the map defined by*

$$q(L_+, L_-) = L_- L_+^{-1}.$$

*Then*

$$q(g \circ (L_+, L_-)) = g^{-1} L_- L_+^{-1} g.$$

The notion of Poisson group actions may be generalized as follows. Let $A \times M \to M$ be a Poisson group action of a Poisson–Lie group $A$ on a Poisson

manifold $M$. A subgroup $K \subset A$ is called *admissible* if the set $C^\infty(M)^K$ of $K$-invariants is a Poisson subalgebra in $C^\infty(M)$. If space $M/K$ is a smooth manifold, we may identify the algebras $C^\infty(M/K)$ and $C^\infty(M)^K$. Hence there exists a Poisson structure on $M/K$ such that the canonical projection $M \to M/K$ is a Poisson map.

**Proposition 4.2.3 ([24], Theorem 6; [20], §2)** *Let $(\mathfrak{a}, \mathfrak{a}^*)$ be the tangent Lie bialgebra of $A$. A connected Lie subgroup $K \subset A$ with Lie algebra $\mathfrak{k} \subset \mathfrak{a}$ is admissible if $\mathfrak{k}^\perp \subset \mathfrak{a}^*$ is a Lie subalgebra.*

We shall need the following particular example of dual pairs arising from Poisson group actions.

Let $A \times M \to M$ be a right Poisson group action of a Poisson–Lie group $A$ on a manifold $M$. Suppose that this action possesses a moment mapping $\mu : M \to A^*$. Let $K$ be an admissible subgroup in $A$. Denote by $\mathfrak{k}$ the Lie algebra of $K$. Assume that $\mathfrak{k}^\perp \subset \mathfrak{a}^*$ is a Lie subalgebra in $\mathfrak{a}^*$. Suppose also that there is a splitting $\mathfrak{a}^* = \mathfrak{t} \oplus \mathfrak{k}^\perp$, and that $\mathfrak{t}$ is a Lie subalgebra in $\mathfrak{a}^*$. Then the linear space $\mathfrak{k}^*$ is naturally identified with $\mathfrak{t}$. Assume that $A^*$ is the semidirect product of the Lie subgroups $K^\perp, T$ corresponding to the Lie algebras $\mathfrak{k}^\perp, \mathfrak{t}$ respectively. Suppose that $K^\perp$ is a connected subgroup in $A^*$. Fix the decomposition $A^* = K^\perp T$ and denote by $\pi_{K^\perp}, \pi_T$ the projections onto $K^\perp$ and $T$ in this decomposition.

**Proposition 4.2.4** *Define a map $\overline{\mu} : M \to T$ by*

$$\overline{\mu} = \pi_T \mu.$$

*Then*

*(i) $\overline{\mu}^*(C^\infty(T))$ is a Poisson subalgebra in $C^\infty(M)$, and hence one can equip $T$ with a Poisson structure such that $\overline{\mu} : M \to T$ is a Poisson map.*

*(ii)Moreover, the algebra $C^\infty(M)^K$ is the centralizer of $\overline{\mu}^*(C^\infty(T))$ in the Poisson algebra $C^\infty(M)$. In particular, if $M/K$ is a smooth manifold the maps*

$$\begin{array}{ccccc} & & M & & \\ & \overset{\pi}{\swarrow} & & \overset{\overline{\mu}}{\searrow} & , \\ M/K & & & & T \end{array} \tag{4.2.3}$$

*form a dual pair.*

*Proof.* (i)First, by Theorem 4.9 in [20] there exists a Poisson bracket on $A^*$ such that $\mu : M \to A^*$ is a Poisson map. Moreover, we can choose this

bracket to be the sum of the standard Poisson–Lie bracket of $A^*$ and of a left invariant bivector on $A^*$. Denote by $A^*_M$ the manifold $A^*$ equipped with this Poisson structure. Now observe that $T$ is identified with the quotient $K^\perp \setminus A^*_M$, where $K^\perp$ acts on $A^*_M$ by multiplications from the left. Therefore to prove part (i) of the proposition it suffices to show that $K^\perp$–invariant functions on $A^*_M$ form a Poisson subalgebra in $C^\infty(A^*_M)$.

Observe that since $A^*$ is a Poisson–Lie group and the Poisson structure of $A^*_M$ is obtained from that of $A^*$ by adding a left–invariant term, the action of $A^*$ on $A^*_M$ by multiplications from the left is a Poisson group action. Note also that $K^\perp$ is a connected subgroup in $A^*$ and $(\mathfrak{k}^\perp)^\perp \cong \mathfrak{k}$ is a Lie subalgebra in $\mathfrak{a}$. Therefore by Proposition 4.2.3 $K^\perp$ is an admissible subgroup in $A^*$. Therefore $K^\perp$–invariant functions on $A^*_M$ form a Poisson subalgebra in $C^\infty(A^*_M)$, and hence $\overline{\mu}^*\left(C^\infty\left(T\right)\right)$ is a Poisson subalgebra in $C^\infty\left(M\right)$. This proves part (i).

(ii)By the definition of the moment map we have:

$$L_{\widehat{X}}\varphi = \langle \mu^*(\theta_{A^*}), X\rangle(\xi_\varphi), \tag{4.2.4}$$

where $X \in \mathfrak{a}, \widehat{X}$ is the corresponding vector field on $M$ and $\xi_\varphi$ is the Hamiltonian vector field of $\varphi \in C^\infty(M)$. Since $A^*$ is the semidirect product of $K^\perp$ and $T$ the pullback of the right–invariant Maurer–Cartan form $\mu^*(\theta_{A^*})$ may be represented as follows:

$$\mu^*(\theta_{A^*}) = \mathrm{Ad}(\pi_{K^\perp}\mu)(\overline{\mu}^*\theta_T) + (\pi_{K^\perp}\mu)^*\theta_{K^\perp},$$

where $\mathrm{Ad}(\pi_{K^\perp}\mu)(\overline{\mu}^*\theta_T) \in \mathfrak{t},\ (\pi_{K^\perp}\mu)^*\theta_{K^\perp} \in \mathfrak{k}^\perp$.

Now let $X \in \mathfrak{k}$. Then $\langle(\pi_{K^\perp}\mu)^*\theta_{K^\perp}), X\rangle = 0$ and formula (4.2.4) takes the form:

$$\begin{gathered} L_{\widehat{X}}\varphi = \langle \mathrm{Ad}(\pi_{K^\perp}\mu)(\overline{\mu}^*\theta_T), X\rangle(\xi_\varphi) = \\ \langle \mathrm{Ad}(\pi_{K^\perp}\mu)(\theta_T), X\rangle(\overline{\mu}_*(\xi_\varphi)). \end{gathered} \tag{4.2.5}$$

Since $\mathrm{Ad}(\pi_{K^\perp}\mu)$ is a non–degenerate transformation, $L_{\widehat{X}}\varphi = 0$ for every $X \in \mathfrak{k}$ if and only if $\overline{\mu}_*(\xi_\varphi) = 0$, i.e. a function $\varphi \in C^\infty(M)$ is $K$–invariant if and only if $\{\varphi, \overline{\mu}^*(\psi)\} = 0$ for every $\psi \in C^\infty(T)$. This completes the proof.

**Remark 4.2.6** *Let $t \in T$ be as in Lemma 4.2.1. Assume that $\pi(\overline{\mu}^{-1}(t))$ is a smooth manifold ($M/K$ does not need to be smooth). Then the algebra $C^\infty(\pi(\overline{\mu}^{-1}(t)))$ is isomorphic to the reduced Poisson algebra $C^\infty(\overline{\mu}^{-1}(t))^{C^\infty(T)}$.*

## 4.3 Quantization of Poisson–Lie groups and Whittaker model

Let $\mathfrak{g}$ be a finite–dimensional complex simple Lie algebra. Observe that cocycle (3.4.6) equips $\mathfrak{g}$ with the structure of a factorizable Lie bialgebra. For simplicity we suppose that $\pi = id$, and denote $s_{id} = s$. Using the identification End $\mathfrak{g} \cong \mathfrak{g} \otimes \mathfrak{g}$ the corresponding r–matrix may be represented as

$$r^s = P_+ - P_- + \frac{1+s}{1-s}P_0,$$

where $P_+, P_-$ and $P_0$ are the projection operators onto $\mathfrak{n}_+, \mathfrak{n}_-$ and $\mathfrak{h}$ in the direct sum

$$\mathfrak{g} = \mathfrak{n}_+ + \mathfrak{h} + \mathfrak{n}_-.$$

Let $G$ be the connected simply connected simple Poisson–Lie group with the tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, $G^*$ the dual group. Observe that $G$ is an algebraic group (see §104, Theorem 12 in [31]).

Note also that

$$r^s_+ = P_+ + \frac{1}{1-s}P_0, \quad r^s_- = -P_- + \frac{s}{1-s}P_0,$$

and hence the subspaces $\mathfrak{b}_\pm$ and $\mathfrak{n}_\pm$ defined by (4.1.3) coincide with the Borel subalgebras in $\mathfrak{g}$ and their nil–radicals, respectively. Therefore every element $(L_+, L_-) \in G^*$ may be uniquely written as

$$(L_+, L_-) = (h_+, h_-)(n_+, n_-), \tag{4.3.1}$$

where $n_\pm \in N_\pm$, $h_+ = exp(\frac{1}{1-s}x)$, $h_- = exp(\frac{s}{1-s}x)$, $x \in \mathfrak{h}$. In particular, $G^*$ is a solvable algebraic subgroup in $G \times G$.

For every algebraic variety $V$ we denote by $\mathcal{F}(V)$ the algebra of regular functions on $V$. Our main object will be the algebra of regular functions on $G^*$, $\mathcal{F}(G^*)$. This algebra may be explicitly described as follows. Let $\pi_V$ be a finite–dimensional representation of $G$. Then matrix elements of $\pi_V(L_\pm)$ are well–defined functions on $G^*$, and $\mathcal{F}(G^*)$ is the subspace in $C^\infty(G^*)$ generated by matrix elements of $\pi_V(L_\pm)$, where $V$ runs through all finite–dimensional representations of $G$.

The elements $L^{\pm,V} = \pi_V(L_\pm)$ may be viewed as elements of the space $\mathcal{F}(G^*) \otimes \text{End}V$. For every two finite–dimensional $\mathfrak{g}$ modules $V$ and $W$ we denote ${r^s_+}^{VW} = (\pi_V \otimes \pi_W)r^s_+$, where $r^s_+$ is regarded as an element of $\mathfrak{g} \otimes \mathfrak{g}$.

**Proposition 4.3.1 ([25], Section 2)** *$\mathcal{F}(G^*)$ is a Poisson subalgebra in the Poisson algebra $C^\infty(G^*)$, the Poisson brackets of the elements $L^{\pm,V}$ are given by*

$$\begin{aligned} \{L_1^{\pm,W}, L_2^{\pm,V}\} &= 2[r_+^{s\ VW}, L_1^{\pm,W} L_2^{\pm,V}], \\ \{L_1^{-,W}, L_2^{+,V}\} &= 2[r_+^{s\ VW}, L_1^{-,W} L_2^{+,V}], \end{aligned} \tag{4.3.2}$$

*where*

$$L_1^{\pm,W} = L^{\pm,W} \otimes I_V, \quad L_2^{\pm,V} = I_W \otimes L^{\pm,V},$$

*and $I_X$ is the unit matrix in $X$.*

*Moreover, the map $\Delta : \mathcal{F}(G^*) \to \mathcal{F}(G^*) \otimes \mathcal{F}(G^*)$ dual to the multiplication in $G^*$,*

$$\Delta(L_{ij}^{\pm,V}) = \sum_k L_{ik}^{\pm,V} \otimes L_{kj}^{\pm,V}, \tag{4.3.3}$$

*is a homomorphism of Poisson algebras.*

**Remark 4.3.7** *Recall that a Poisson–Hopf algebra is a Poisson algebra which is also a Hopf algebra such that the comultiplication is a homomorphism of Poisson algebras. According to Proposition 4.3.2 $\mathcal{F}(G^*)$ is a Poisson–Hopf algebra.*

Now we describe a quantization of the Poisson–Hopf algebra $\mathcal{F}(G^*)$. Let $\tilde{U}_h^s(\mathfrak{g})$ be the subalgebra in $U_h^s(\mathfrak{g})$ topologically generated, in the sense of formal power series over $\mathbb{C}[[h]]$, by elements $\tilde{H}_i = hH_i,\ i = 1, \ldots l,\ \tilde{e}_\beta = he_\beta,\ \tilde{f}_\beta = hf_\beta,\ \beta \in \Delta_+$.

In fact $\tilde{U}_h^{s_\pi}(\mathfrak{g})$ is a Hopf subalgebra in $U_h^s(\mathfrak{g})$, explicit formulas for the comultiplication may be obtained using Proposition 8.3 in [15].

**Proposition 4.3.2** *$\tilde{U}_h^s(\mathfrak{g})$ is a quantum formal series Hopf algebra (or QFSH algebra), i.e. $\tilde{U}_h^s(\mathfrak{g})$ is isomorphic as a $\mathbb{C}[[h]]$–module to $Map(I, \mathbb{C}[[h]])$ for some set $I$, and $\tilde{U}^s(\mathfrak{g}) = \tilde{U}_h^s(\mathfrak{g})/h\tilde{U}_h^s(\mathfrak{g}) \cong \mathbb{C}[[\xi_1, \xi_2, \ldots]]$ as a topological algebra, for some (possibly infinite) sequence of indeterminates $\xi_1, \xi_2, \ldots$.*

*Proof* is similar to the proof of the same result for $U_h(\mathfrak{g})$ (see Section 8.3 C in [6]).

Note that $\tilde{U}^s(\mathfrak{g})$ is naturally a Poisson–Hopf algebra, the Poisson bracket is given by

$$\{x_1, x_2\} = \frac{[a_1, a_2]}{h} \ (\mathrm{mod}\ h), \tag{4.3.4}$$

if $a_1, a_2 \in \tilde{U}_h^s(\mathfrak{g})$ reduce to $x_1, x_2 \in \tilde{U}^s(\mathfrak{g})$ (mod $h$).

For any finite–dimensional $U_h^s(\mathfrak{g})$ module $V[[h]]$ we denote by ${}^hL^{\pm,V}$ the following elements of $U_h^s(\mathfrak{g}) \otimes \mathrm{End}V[[h]]$ (see [11]):

$$ {}^hL^{+,V} = (id \otimes \pi_V){\mathcal{R}_{21}^s}^{-1} = (id \otimes \pi_V S^s)\mathcal{R}_{21}^s, \quad {}^hL^{-,V} = (id \otimes \pi_V)\mathcal{R}^s. $$

We also denote $R^{VW} = (\pi_V \otimes \pi_W)\mathcal{R}^s$. Observe that from formula (3.4.4) it follows that actually ${}^hL^{\pm,V} \in \tilde{U}_h^s(\mathfrak{g}) \otimes \mathrm{End}V[[h]]$. If we fix a basis in $V[[h]]$, ${}^hL^{\pm,V}$ may be regarded as matrices with matrix elements $({}^hL^{\pm,V})_{ij}$ being elements of $\tilde{U}_h^s(\mathfrak{g})$. From the Yang–Baxter equation for $\mathcal{R}$ we get relations between $L^{\pm,V}$:

$$ R^{VW}\,{}^hL_1^{\pm,W}\,{}^hL_2^{\pm,V} = {}^hL_2^{\pm,V}\,{}^hL_1^{\pm,W}R^{VW}, \qquad (4.3.5) $$

$$ R^{VW}\,{}^hL_1^{-,W}\,{}^hL_2^{+,V} = {}^hL_2^{+,V}\,{}^hL_1^{-,W}R^{VW}. \qquad (4.3.6) $$

By ${}^hL_1^{\pm,W}$, ${}^hL_2^{\pm,V}$ we understand the following matrices in $V \otimes W$:

$$ {}^hL_1^{\pm,W} = {}^hL^{\pm,W} \otimes I_V, \quad {}^hL_2^{\pm,V} = I_W \otimes {}^hL^{\pm,V}, $$

where $I_X$ is the unit matrix in $X$.

From (3.1.5) we can obtain the action of the comultiplication on the matrices ${}^hL^{\pm,V}$:

$$ \Delta_s({}^hL_{ij}^{\pm,V}) = \sum_k {}^hL_{ik}^{\pm,V} \otimes {}^hL_{kj}^{\pm,V}. \qquad (4.3.7) $$

We denote by $\mathcal{F}_h(G^*)$ the Hopf subalgebra in $\tilde{U}_h^s(\mathfrak{g})$ generated in the sense of $h$–adic topology by matrix elements of ${}^hL^{\pm,V}$, where $V$ runs through all finite–dimensional representations of $\mathfrak{g}$.

**Proposition 4.3.3** *Denote by $p : \tilde{U}_h^s(\mathfrak{g}) \to \tilde{U}^s(\mathfrak{g})$ the canonical projection. Then $p(\mathcal{F}_h(G^*))$ is isomorphic to $\mathcal{F}(G^*)$ as a Poisson–Hopf algebra.*

*Proof.* Denote $\mathcal{F}(G^*)' = p(\mathcal{F}_h(G^*))$, $\tilde{L}^{\pm,V} = p({}^hL^{\pm,V}) \in \mathcal{F}(G^*)' \otimes \mathrm{End}V$. First observe that the map

$$ \imath : \mathcal{F}(G^*)' \to \mathcal{F}(G^*), \quad (\imath \otimes id)\tilde{L}^{\pm,V} = L^{\pm,V} $$

is a well–defined linear isomorphism. Indeed, consider, for instance, element $\tilde{L}^{-,V}$. From (3.4.4) it follows that

$$ \begin{gathered} \tilde{L}_{ij}^{-,V} = \{exp\left[\sum_{i=1}^l -2p(hY_i) \otimes \pi_V(\tfrac{s}{1-s}H_i)\right] \times \\ \prod_\beta exp[2(X_\beta, X_{-\beta})^{-1}p(he_\beta) \otimes \pi_V(X_{-\beta})]\}_{ij}. \end{gathered} \qquad (4.3.8) $$

On the other hand (4.3.1) implies that every element $L_-$ may be represented in the form

$$\begin{array}{l} L_- = exp\left[\sum_{i=1}^l b_i \frac{s}{1-s} H_i\right] \times \\ \prod_\beta exp[b_\beta X_{-\beta}],\ b_i, b_\beta \in \mathbb{C}, \end{array} \tag{4.3.9}$$

and hence

$$\begin{array}{l} L_{ij}^{-,V} = \{exp\left[\sum_{i=1}^l b_i \otimes \pi_V(\frac{s}{1-s} H_i)\right] \times \\ \prod_\beta exp[b_\beta \otimes \pi_V(X_{-\beta})]\}_{ij}. \end{array} \tag{4.3.10}$$

Therefore $\imath$ is a linear isomorphism. We have to prove that $\imath$ is an isomorphism of Poisson–Hopf algebras.

Recall that $\mathcal{R}^s = 1 \otimes 1 + 2hr_+^s \pmod{h^2}$. Therefore from commutation relations (4.3.5), (4.3.6) it follows that $\mathcal{F}(G^*)'$ is a commutative algebra, and the Poisson brackets of matrix elements $\tilde{L}_{ij}^{\pm,V}$ (see (4.3.4)) are given by (4.3.2), where $L^{\pm,V}$ are replaced by $\tilde{L}^{\pm,V}$. From (4.3.7) we also obtain that the action of the comultiplication on the matrices $\tilde{L}^{\pm,V}$ is given by (4.3.3), where $L^{\pm,V}$ are replaced by $\tilde{L}^{\pm,V}$. This completes the proof.

We shall call the map $p : \mathcal{F}_h(G^*) \to \mathcal{F}(G^*)$ the quasiclassical limit.

Now using the Hopf algebra $\mathcal{F}_h(G^*)$ we shall define another quantum version of the Whittaker model $W(\mathfrak{b}_-)$. Let $\mathcal{F}_h(N_\pm)$ be the subalgebras in $\mathcal{F}_h(G^*)$ generated by matrix elements of the matrices $N^{-,V} = (id \otimes \pi_V)\mathcal{R}_\Delta^s$, $N^{+,V} = (id \otimes \pi_V)\mathcal{R}_{\Delta 21}^{s\ -1}$, where

$$\mathcal{R}_\Delta^s = \prod_\beta exp_{q_\beta^{-1}}[(q-q^{-1})a(\beta)^{-1} e_\beta \otimes e^{h\frac{1+s}{1-s}\beta^\vee} f_\beta].$$

Suppose that the ordering of the root system $\Delta_+$ is fixed as in Proposition 3.2.6. Then by Proposition 3.2.6 the map $\chi_h^s : \mathcal{F}_h(N_-) \to \mathbb{C}$ defined by

$$(\chi_h^s \otimes id)(N^{-,V}) = \prod_{i=1}^l exp_{q_{\alpha_i}^{-1}}[\frac{(q_i - q_i^{-1})}{h} c_i \otimes \pi_V(e^{hd_i\frac{1+s}{1-s}H_i} f_i)], c_i \in \mathbb{C}[[h]],\ c_i \neq 0 \tag{4.3.11}$$

is a character of $\mathcal{F}_h(N_-)$.

We also denote by $\mathcal{F}_h(H)$ the intersection $U_h^s(\mathfrak{h}) \cap \mathcal{F}_h(G^*)$. Clearly, $\mathcal{F}_h(H)$ is a commutative subalgebra in $\mathcal{F}_h(G^*)$. From commutation relations (4.3.6) one can obtain the following weak version of the Poincaré–Birkhoff–Witt theorem for $\mathcal{F}_h(G^*)$.

**Proposition 4.3.4** *Multiplication defines an isomorphism of* $\mathbb{C}[[h]]$*–modules*

$$\mathcal{F}_h(N_+) \otimes \mathcal{F}_h(H) \otimes \mathcal{F}_h(N_-) \to \mathcal{F}_h(G^*).$$

Define $\mathcal{F}_h(B_\pm) = \mathcal{F}_h(N_\pm)\mathcal{F}_h(H)$. Let $\mathcal{F}_h(N_-)_{\chi_h^s}$ be the kernel of the character $\chi_h^s$ so that one has a direct sum

$$\mathcal{F}_h(N_-) = \mathbb{C}[[h]] \oplus \mathcal{F}_h(N_-)_{\chi_h^s}. \tag{4.3.12}$$

From Proposition 4.3.4 and formula (4.3.12) we obtain also the direct sum

$$\mathcal{F}_h(G^*) = \mathcal{F}_h(B_+) \oplus I_{\chi_h^s}, \tag{4.3.13}$$

where $I_{\chi_h^s} = \mathcal{F}_h(G^*)\mathcal{F}_h(N_-)_{\chi_h^s}$ is the left–sided ideal generated by $\mathcal{F}_h(N_-)_{\chi_h^s}$.

Denote by $\rho_{\chi_h^s}$ the projection onto $\mathcal{F}_h(B_+)$ in the direct sum (4.3.13). Let $Z(\mathcal{F}_h(G^*))$ be the center of $\mathcal{F}_h(G^*)$. Similarly to the classical case we define a subspace $W_h(B_+)$ in $\mathcal{F}_h(B_+)$ by $W_h(B_+) = \rho_{\chi_h^s}(Z(\mathcal{F}_h(G^*)))$.

To formulate the quantum version of Theorem A for $W_h(B_+)$ we recall that for any finite–dimensional $\mathfrak{g}$–module $V$ the element

$$C_V = (id \otimes tr_V)((S^s \otimes id)(L^{+,V})L^{-,V}(1 \otimes e^{2h\rho^\vee})),$$

where $tr_V$ is the trace in $V[[h]]$, is central in $\mathcal{F}_h(G^*)$ (see formulas (3.5.1) and (3.1.7)).

**Theorem $\mathbf{A_q}$** *(i)The map*

$$\rho_{\chi_h^{s\pi}} : Z(\mathcal{F}_h(G^*)) \to W_h(B_+) \tag{4.3.14}$$

*is an isomorphism of algebras.*

*(ii) The algebra* $W_h(B_+)$ *is freely generated as a commutative topological algebra over* $\mathbb{C}[[h]]$ *by the elements* $C_{V_i}^{\rho_{\chi_h^s}} = \rho_{\chi_h^s}(C_{V_i}),\ i = 1, \ldots, l$, *where* $V_i,\ i = 1, \ldots l$ *are the fundamental representations of* $\mathfrak{g}$.

*Proof* of (i) is similar to that of Theorem A in the classical case. Part (ii) will be proved in Section 4.5.

**Corollary 4.3.5** *The algebra* $Z(\mathcal{F}_h(G^*))$ *is freely generated as a commutative topological algebra over* $\mathbb{C}[[h]]$ *by the elements* $C_{V_i}$, *where* $V_i,\ i = 1, \ldots l$ *are the fundamental representations of* $\mathfrak{g}$.

**Definition $\mathbf{A_q}$** *The algebra*

$$W_h(B_+) = \rho_{\chi_h^s}(Z(\mathcal{F}_h(G^*))).$$

*is called the Whittaker model of the center $Z(\mathcal{F}_h(G^*))$.*

Now following Section 1.2 (see Lemma A) we equip $\mathcal{F}_h(B_+)$ with a structure of a left $\mathcal{F}_h(N_-)$ module in such a way that $W_h(B_+)$ is realized as the space of invariants with respect to this action. For every $v \in \mathcal{F}_h(B_+)$ and $x \in \mathcal{F}_h(N_-)$ we put

$$x \cdot v = \rho_{\chi_h^s}([x, v]). \tag{4.3.15}$$

Consider the space $\mathcal{F}_h(B_+)^{\mathcal{F}_h(N_-)}$ of $\mathcal{F}_h(N_-)$ invariants in $\mathcal{F}_h(B_+)$ with respect to this action. Clearly, $W_h(B_+) \subseteq \mathcal{F}_h(B_+)^{\mathcal{F}_h(N_-)}$.

To formulate the quantum version of Theorem B for $W_h(B_+)$ we have to impose a restriction on the coefficients $c_i$ in (4.3.11). Define an element $u \in N_-$ by

$$u = \prod_{i=1}^{l} exp[2d_i c_i^0 X_{-\alpha_i}],\ c_i^0 = c_i \ (\mathrm{mod}\ h), \tag{4.3.16}$$

where the terms in the product are ordered as in (4.3.11). The motivation for this definition will be explained in the next section.

**Theorem $\mathbf{B_q}$** *Suppose that $u \in N_+ s N_+ \cap N_-$, where $s$ stands for a representative of the Coxeter element in $G$. Then the space of $\mathcal{F}_h(N_-)$ invariants in $\mathcal{F}_h(B_+)$ with respect to the action (4.3.15) is isomorphic to $W_h(B_+)$, i.e.*

$$\mathcal{F}_h(B_+)^{\mathcal{F}_h(N_-)} \cong W_h(B_+). \tag{4.3.17}$$

The proof of this theorem occupies two next sections.

**Remark 4.3.8** *The following lemma shows that the set $N_+ s N_+ \cap N_-$ is not empty.*

***Lemma 4.3.6* ([29], Lemma 4.5)** *Let $w_0 \in W$ be the longest element; let $\tau \in Aut\ \Delta_+$ be the automorphism defined by $\tau(\alpha) = -w_0\alpha, \alpha \in \Delta_+$. Let $N_i \subset N_+$ be the 1-parameter subgroup generated by the root vector $X_{\tau(\alpha_i)}, i = 1, \ldots l$. Choose an element $u_i \in N_i, u_i \neq 1$. Then we have $w_0 u_i w_0^{-1} \in B_+ s_i B_+$. We may fix $u_i$ in such a way that $w_0 u_i w_0^{-1} \in N_+ s_i N_+$. Set $x = u_1 u_2 ... u_l$. Then $f = w_0 x w_0^{-1} \in N_+ s N_+ \cap N_-$.*

Similarly to Proposition 2.6.1 we also have the following homological description of $W_h(B_+)$.

**Proposition 4.3.7** *Suppose that the conditions of Theorem $B_q$ are satisfied. Then $W_h(B_+)$ is isomorphic to $Hk^0(\mathcal{F}_h(G^*),\mathcal{F}_h(N_-),\chi_h^s)^{opp}$ as an associative algebra.*

## 4.4 Poisson reduction and the Whittaker model

In this section we start the proof of Theorem $\mathrm{B}_q$. We shall analyse the quasiclassical limit of the algebra $\mathcal{F}_h(B_+)^{\mathcal{F}_h(N_-)}$. Using results of Section 4.2 we realize this limit algebra as the algebra of functions on a reduced Poisson manifold.

Denote $\mathcal{F}(N_\pm)=p(\mathcal{F}_h(N_\pm)),\ \mathcal{F}(B_\pm)=p(\mathcal{F}_h(B_\pm)),\ \mathcal{F}(H)=p(\mathcal{F}_h(H))$. We denote by $\chi_h^s$ the character of the Poisson subalgebra $\mathcal{F}(N_-)$ such that $\chi^s(p(x))=\chi_h^s(x) \pmod h$ for every $x\in\mathcal{F}_h(N_-)$. From (4.3.11) we have

$$(\chi^s\otimes id)p(N^{-,V})=\prod_{i=1}^{l}exp[2d_ic_i^0\otimes\pi_V(X_{-\alpha_i})],\quad c_i^0=c_i \pmod h. \tag{4.4.1}$$

Let $\mathcal{F}(N_-)_{\chi^s}$ be the kernel of the character $\chi^s$ so that one has a direct sum

$$\mathcal{F}(N_-)=\mathbb{C}\oplus\mathcal{F}(N_-)_{\chi^s}. \tag{4.4.2}$$

Similarly to (4.3.13) we have the direct sum

$$\mathcal{F}(G^*)=\mathcal{F}(B_+)\oplus I_{\chi^s}, \tag{4.4.3}$$

where $I_{\chi^s}=\mathcal{F}(G^*)\mathcal{F}(N_-)_{\chi^s}$ is the left–sided ideal generated by $\mathcal{F}(N_-)_{\chi^s}$.

Denote by $\rho_{\chi^s}$ the projection onto $\mathcal{F}(B_+)$ in the direct sum (4.4.3). Using Lemma $\mathrm{A}_q$ we define the quasiclassical limit of action (4.3.15) by

$$x\cdot v=\rho_{\chi^s}(\{x,v\}), \tag{4.4.4}$$

where $v\in\mathcal{F}(B_+)$ and $x\in\mathcal{F}(N_-)$. We shall describe the space of invariants $\mathcal{F}(B_+)^{\mathcal{F}(N_-)}$ with respect to this action by analysing "dual geometric objects".

First observe that algebra $\mathcal{F}(B_+)^{\mathcal{F}(N_-)}$ is a particular example of the reduced Poisson algebra introduced in Lemma 4.2.1. Indeed, define a map $\mu_{N_-}:G^*\to N_-$ by

$$\mu_{N_+}(L_+,L_-)=n_-, \tag{4.4.5}$$

where $n_-$ is given by (4.3.1). $\mu_{N_-}$ is a morphism of algebraic varieties. We also note that by definition $\mathcal{F}(N_-) = \{\varphi \in \mathcal{F}(G^*) : \varphi = \varphi(n_-)\}$. Therefore $\mathcal{F}(N_-)$ is generated by the pullbacks of regular functions on $N_-$. Since $\mathcal{F}(N_-)$ is a Poisson subalgebra in $\mathcal{F}(G^*)$, and regular functions on $N_-$ are dense in $C^\infty(N_-)$ on every compact subset, we can equip the manifold $N_-$ with the Poisson structure in such a way that $\mu_{N_+}$ becomes a Poisson mapping. Let $u$ be the element defined by (4.3.16),

$$u = \prod_{i=1}^{l} exp[2d_i c_i^0 X_{-\alpha_i}] \ \in N_-. \tag{4.4.6}$$

Then from (4.4.1) it follows that $\chi^s(\varphi) = \varphi(u)$ for every $\varphi \in \mathcal{F}(N_-)$. $\chi^s$ naturally extends to a character of the Poisson algebra $C^\infty(N_-)$.

Now applying Lemma 4.2.1 for $M = G^*,\ B = N_-,\ \pi = \mu_{N_+},\ b = u$ we can define the reduced Poisson algebra $C^\infty(\mu_{N_+}^{-1}(u))^{C^\infty(N_-)}$ (see also Remark 4.2.5). Denote by $I_u$ the ideal in $C^\infty(G^*)$ generated by elements $\mu_{N_+}^*\psi,\ \psi \in C^\infty(N_-),\ \psi(u) = 0$. Let $P_u : C^\infty(G^*) \to C^\infty(G^*)/I_u = C^\infty(\mu_{N_+}^{-1}(u))$ be the canonical projection. Then the action (4.2.1) of $C^\infty(N_-)$ on $C^\infty(\mu_{N_+}^{-1}(u))$ takes the form:

$$\psi \cdot \varphi = P_u(\{\mu_{N_+}^*\psi, \tilde{\varphi}\}), \tag{4.4.7}$$

where $\psi \in C^\infty(N_-),\ \varphi \in C^\infty(\mu_{N_+}^{-1}(u))$ and $\tilde{\varphi} \in C^\infty(G^*)$ is a representative of $\varphi$ such that $P_u\tilde{\varphi} = \varphi$.

**Lemma 4.4.1** *$\mu_{N_+}^{-1}(u)$ is a subvariety in $G^*$. Moreover, the algebra $\mathcal{F}(B_+)^{\mathcal{F}(N_-)}$ is isomorphic to the algebra of regular functions on $\mu_{N_+}^{-1}(u)$ which are invariant with respect to the action (4.4.7) of $C^\infty(N_-)$ on $C^\infty(\mu_{N_+}^{-1}(u))$, i.e.*

$$\mathcal{F}(B_+)^{\mathcal{F}(N_-)} = \mathcal{F}(\mu_{N_+}^{-1}(u)) \cap C^\infty(\mu_{N_+}^{-1}(u))^{C^\infty(N_-)}.$$

*Proof.* By definition $\mu_{N_+}^{-1}(u)$ is a subvariety in $G^*$. Next observe that $I_{\chi^s} = \mathcal{F}(G^*) \cap I_u$. Therefore the algebra $\mathcal{F}(B_+) = \mathcal{F}(G^*)/I_{\chi^s}$ is identified with the algebra of regular functions on $\mu_{N_+}^{-1}(u)$.

Since $\mathcal{F}(N_-)$ is dense in $C^\infty(N_-)$ on every compact subset in $N_-$ we have:

$$C^\infty(\mu_{N_+}^{-1}(u))^{C^\infty(N_-)} \cong C^\infty(\mu_{N_+}^{-1}(u))^{\mathcal{F}(N_-)}.$$

Finally observe that action (4.4.7) coincides with action (4.4.4) when restricted to regular functions.

We shall realize the algebra $C^\infty(\mu_{N_+}^{-1}(u))^{C^\infty(N_-)}$ as the algebra of functions on a reduced Poisson manifold. In the spirit of Lemma 4.2.1 we shall construct a map that forms a dual pair together with the mapping $\mu_{N_+}$. In this construction we use the dressing action of the Poisson–Lie group $G$ on $G^*$ (see Proposition 4.2.2).

Consider the restriction of the dressing action $G^* \times G \to G^*$ to the subgroup $N_+ \subset G$. Note that by Proposition 4.1.2 (i), (iii) and Proposition 4.2.3 $N_+$ is an admissible subgroup in $G$. Therefore $C^\infty(G^*)^{N_+}$ is a subalgebra in the Poisson algebra $C^\infty(G^*)$.

**Proposition 4.4.2** *The algebra $C^\infty(G^*)^{N_+}$ is the centralizer of $\mu_{N_+}^*(C^\infty(N_-))$ in the Poisson algebra $C^\infty(G^*)$.*

*Proof.* We shall prove the proposition in a few steps. First we restrict the dressing action of $G$ on $G^*$ the the Borel subgroup $B_+$. According to part (iii) of Proposition 4.1.2 $(\mathfrak{b}_+, \mathfrak{b}_-)$ is a subbialgebra of $(\mathfrak{g}, \mathfrak{g}^*)$. Therefore $B_+$ is a Poisson–Lie subgroup in $G$.

By Proposition 4.2.2 for $X \in \mathfrak{b}_+$ we have:

$$L_{\widehat{X}}\varphi(L_+, L_-) = (\theta_{G^*}(L_+, L_-), X)(\xi_\varphi) = (r_-^{-1}\mu_{B_+}^*(\theta_{B_-}), X)(\xi_\varphi), \quad (4.4.8)$$

where $\widehat{X}$ is the corresponding vector field on $G^*$, $\xi_\varphi$ is the Hamiltonian vector field of $\varphi \in C^\infty(G^*)$, and the map $\mu_{B_+} : G^* \to B_-$ is defined by $\mu_{B_+}(L_+, L_-) = L_-$. Now from Proposition 4.1.2 (iv) and the definition of the moment map it follows that $\mu_{B_+}$ is a moment map for the dressing action of the subgroup $B_+$ on $G^*$.

Observe that the orthogonal complement of the Lie subalgebra $\mathfrak{n}_+ \subset \mathfrak{b}_+$ in the dual space $\mathfrak{b}_-$ coincides with the Lie subalgebra $\mathfrak{h} \subset \mathfrak{b}_-$. Hence by Proposition 4.2.3 $N_+$ is an admissible subgroup in the Lie–Poisson group $B_+$. Moreover the dual group $B_-$ is the semidirect product of the Lie groups $H$ and $N_-$ corresponding to the Lie algebras $\mathfrak{n}_+^\perp = \mathfrak{h}$ and $\mathfrak{n}_+^* = \mathfrak{n}_-$ , respectively. We conclude that all the conditions of Proposition 4.2.4 are satisfied with $A = B_+, K = N_+, A^* = B_-, T = N_-, K^\perp = H, \mu = \mu_{B_+}$. It follows that the algebra $C^\infty(G^*)^{N_+}$ is the centralizer of $\mu_{N_+}^*(C^\infty(N_-))$ in the Poisson algebra $C^\infty(G^*)$. This completes the proof.

Let $G^*/N_+$ be the quotient of $G^*$ with respect to the dressing action of $N_+$, $\pi : G^* \to G^*/N_+$ the canonical projection. Note that the space $G^*/N_+$ is not a smooth manifold. However, in the next section we will see that the subspace $\pi(\mu_{N_+}^{-1}(u)) \subset G^*/N_+$ is a smooth manifold. Therefore by Remark 4.2.6 the algebra $C^\infty(\pi(\mu_{N_+}^{-1}(u)))$ is isomorphic to $C^\infty(\mu_{N_+}^{-1}(u))^{C^\infty(N_-)}$.

Moreover we will see that $\pi(\mu_{N_+}^{-1}(u))$ has a structure of algebraic variety. Using Lemma 4.4.1 we will obtain that the algebra $\mathcal{F}(B_+)^{\mathcal{F}(N_-)}$ is the algebra of regular functions on this variety.

## 4.5 Cross–section theorem

In this section we describe the reduced space $\pi(\mu_{N_+}^{-1}(u)) \subset G^*/N_+$ and the algebra $\mathcal{F}(B_+)^{\mathcal{F}(N_-)}$. We also complete the proof of Theorem $\mathrm{B}_q$.

First observe that using the embedding $q : G^* \to G$ (see Proposition 4.2.2) one can reduce the study of the dressing action to the study of the action of $G$ on itself by conjugations. This simplifies many geometric problems. In particular, consider the restriction of this action to the subgroup $N_+$. Denote by $\pi_q : G \to G/N_+$ the canonical projection onto the quotient with respect to this action. Then we can identify the reduced space $\pi(\mu_{N_+}^{-1}(u))$ with the subspace $\pi_q(q(\mu_{N_+}^{-1}(u)))$ in $G/N_+$. Using this identification we shall explicitly describe the reduced space $\pi(\mu_{N_+}^{-1}(u))$. We start with description of the image of the "level surface" $\mu_{N_+}^{-1}(u)$ under the embedding $q$.

**Proposition 4.5.1** *Let $q : G^* \to G$ be the map introduced in Proposition 4.2.2,*

$$q(L_+, L_-) = L_- L_+^{-1}.$$

*Then $q(\mu_{N_+}^{-1}(u))$ is a subvariety in $N_+ s N_+$.*

*Proof.* First, using definition (4.4.5) of the map $\mu_{N_+}$ we can describe the space $\mu_{N_+}^{-1}(u)$ as follows:

$$\mu_{N_+}^{-1}(u) = \{(h_+ n_+, s(h_+)u) | n_+ \in N_+, h_+ \in H\}, \qquad (4.5.1)$$

since by (4.3.1) $h_- = s(h_+)$. Therefore

$$q(\mu_{N_+}^{-1}(u)) = \{s(h_+) u n_+^{-1} h_+^{-1} | n_+ \in N_+, h_+ \in H\}. \qquad (4.5.2)$$

Now recall that $u \in N_+ s N_+ \cap N_-$, and hence

$$u n_+^{-1} \in N_+ s N_+. \qquad (4.5.3)$$

Next, the space $N_+ s N_+$ is invariant with respect to the following action of $H$:

$$h \circ L = s(h) L h^{-1}. \qquad (4.5.4)$$

Indeed, let $L = vsu,\ v, u \in N_+$ be an element of $N_+sN_+$. Then

$$h \circ L = s(h)vs(h)^{-1}s(h)sh^{-1}huh^{-1} = s(h)vs(h)^{-1}shuh^{-1}. \qquad (4.5.5)$$

The r.h.s. of the last equality belongs to $N_+sN_+$ because $H$ normalizes $N_+$.

Comparing action (4.5.4) with (4.5.2) and adding (4.5.3) we obtain that $q(\mu_{N_+}^{-1}(f)) \subset N_+sN_+$. Since $q$ is an embedding, $q(\mu_{N_+}^{-1}(f))$ is a subvariety in $N_+sN_+$. This concludes the proof.

We identify $\mu_{N_+}^{-1}(u)$ with the subvariety in $N_+sN_+$ described in the previous proposition. As we observed in the beginning of this section the reduced space $\pi(\mu_{N_+}^{-1}(u))$ is isomorphic to $\pi_q(q(\mu_{N_+}^{-1}(u)))$. Note that by Proposition 4.5.1 $q(\mu_{N_+}^{-1}(u)) \subset N_+sN_+$. But the variety $N_+sN_+$ is stable under the action of $N_+$ by conjugations. Therefore to describe the reduced space $\pi_q(q(\mu_{N_+}^{-1}(u)))$ we have to study the structure of the quotient $N_+sN_+/N_+$. Our main geometric result is

**Theorem $\mathbf{C_q}$ ([26], Theorem 3.1)** *Let $N_+' = \{v \in N_+ | s^{-1}(v) \in N_-\}$. Then the action of $N_+$ on $N_+sN_+$ by conjugations is free, and $N_+'s$ is a cross–section for this action, i.e. for each $L \in N_+sN_+$ there exists a unique element $n \in N_+$ such that $nLn^{-1} \in N_+'s$. Moreover, the projection $\pi_q : N_+sN_+ \to N_+'s$ is a morphism of varieties.*

**Lemma 4.5.2 ([5], Theorem 8.4.3, [29] Lemma 7.2)** *$N_+' \subset N$ is an abelian subgroup,* $\dim N_+' = l$. *Moreover, every element $L \in N_+sN_+$ may be uniquely represented in the form $L = vsu, v \in N_+', u \in N_+$.*

*Proof of Theorem $C_q$.* Denote by $h$ the Coxeter number of $\mathfrak{g}$. By definition $h$ is the order of the Coxeter element, $s^h = id$. Note that $h = \frac{2N}{l}$.

Let $C_s \subset W$ be the cyclic subgroup generated by the Coxeter element. $C_s$ has exactly $l$ different orbits in $\Delta$. The proof depends on the structure of these orbits. For this reason we have to distinguish several cases[1].

1.Let $\mathfrak{g}$ be of type $A_l$.

The following lemma is checked by straightforward calculation.

**Lemma 4.5.3** *(i) Each orbit of $C_s$ in $\Delta$ consists of exactly $h$ elements. One can order these orbits in such a way that $k$-th orbit contains all positive roots of height $k$ and all negative roots of height $h-k$.*

Put

$$\mathfrak{n}_k = \bigoplus_{\{\alpha \in \Delta_+,\ ht\,\alpha = k\}} \mathfrak{n}_\alpha, N_k = \exp \mathfrak{n}_k.$$

[1] The proofs given below do not apply when $\mathfrak{g}$ is the simple Lie algebra of type $E_6$.

For each $k$ we can choose $\gamma_k \in \Delta_+$ in such a way that

$$\mathfrak{n}_k = \bigoplus_{p=0}^{h-k-1} \mathfrak{n}_{s^{-p}(\gamma_k)}.$$

Put $\mathfrak{n}_k^p = \mathfrak{n}_{s^{-p}(\gamma_k)}, N_k^p = \exp \mathfrak{n}_k^p$.

Let $L = vsu, v \in N'_+, u \in N_+$. We must find $n \in N_+$ such that

$$nvsu = v_0 sn,\ v_0 \in N'_+. \tag{4.5.6}$$

For any $n \in N_+$ there exists a factorization

$$n = n_1 n_2 \dots n_l,\ \text{where } n_k \in N_k.$$

Moreover, each $n_k$ may be factorized as

$$n_k = n_k^0 n_k^1 \dots n_k^{h-k-1},\ n_k^p \in N_k^p.$$

For any $n \in N_+$ the element $nvsu$ admits a representation

$$nvsu = \tilde{v} s \tilde{u},\ \tilde{v} \in N'_+,\ \tilde{u} \in N_+.$$

Let

$$\tilde{u} = \overrightarrow{\prod_{k=1}^{l}} \overrightarrow{\prod_{p=0}^{h-k-1}} \tilde{u}_k^p,\ \tilde{u}_k^p \in N_k^p,$$

be the corresponding factorization of $\tilde{u}$.

**Lemma 4.5.4** *We have $\tilde{u}_k^p = s^{-1}\left(n_k^{p-1}\right) V_k^p$, where the factors $V_k^p \in N_k^p$ depend only on $u, v$ and on $n_j^q$ with $j < k$.*

Assume now that $n$ satisfies (4.5.6). Then we have $\tilde{v} = v_0, \tilde{u} = n$. This leads to the following relations:

$$s^{-1}\left(n_k^{p-1}\right) V_k^p = n_k^p, \tag{4.5.7}$$

where we set formally $n_k^{-1} = 1$.

**Lemma 4.5.5** *The system (4.5.7) may be solved recursively starting with $k = 1,\ p = 0$.*

Clearly, the solution is unique. This concludes the proof for $\mathfrak{g}$ of type $A_l$.

2. Let now $\mathfrak{g}$ be a simple Lie algebra of type other than $A_l$ and $E_6$.

**Lemma 4.5.6** *(i) The Coxeter number $h$ is even.*

*(ii) Each orbit of $C_s$ in $\Delta$ consists of exactly $h$ elements and contains an equal number of positive and negative roots.*

*(iii) Put*

$$\Delta_+^p = \{\alpha \in \Delta_+; s^p\alpha \notin \Delta_+\},\ \mathfrak{n}^p = \bigoplus_{\alpha\in\Delta_+^p} \mathbb{C}\cdot X_\alpha;$$

*then $\mathfrak{n}^p \subset \mathfrak{n}$ is an abelian subalgebra,* $\dim \mathfrak{n}^p = l$.

If $\mathfrak{g}$ is not of type $D_{2k+1}$ this assertion follows from Proposition 33, Chap.6, no. 1.11 and Corollary 3, Chap.5, no. 6.2 in [1]. For $\mathfrak{g}$ of type $D_{2k+1}$ it may be checked directly.

Put $N^p = \exp \mathfrak{n}^p$. Let $N^p$ be the corresponding subgroup of $G$. Let $L = vsu,\ v \in N'_+,\ u \in N_+$. We must find $n \in N_+$ such that

$$vsu = nv_0sn^{-1},\ v_0 \in N'_+.$$

Put

$$n = n_1n_2\dots n_{\frac{h}{2}},\quad n_p \in N_p. \tag{4.5.8}$$

The elements $n_p$ will be determined recursively. We have

$$vs\,(u) = \overrightarrow{\prod_p} n_pv_0s\left(\overleftarrow{\prod_p} n_p^{-1}\right). \tag{4.5.9}$$

We shall say that an element $x \in G$ is in the big cell in $G$ if $x \in B_+N_- \subset G$.

**Lemma 4.5.7** *$vs\,(u)$ is in the big cell in $G$ and admits a factorization*

$$vs\,(u) = x_+^1x_-^1,\ x_+^1 \in N_+,\ x_-^1 \in N_-.$$

Indeed, let $u = u_{h/2}u_{h/2-1}\dots u_1,\ u_p \in N^p$ be a similar decomposition of $u$. Then we have simply $x_- = s\,(u_1)$. (It is clear that $x_+^1 \in B_+$ actually does not have an $H$-component and so belongs to $N_+$

A comparison of the r.h.s in (4.5.9) with the Bruhat decomposition of the l.h.s. immediately yields that the first factor in (4.5.8) is given by $n_1 = s^{-1}\,(x_-)^{-1}$.

Assume that $n_1, n_2, \ldots, n_{k-1}$ are already computed. Put

$$m_k = n_1 n_2 \ldots n_{k-1}$$

and consider the element

$$L^k := s^{k-1}\left(m_k^{-1} v s(u) s(m_k)\right). \tag{4.5.10}$$

**Lemma 4.5.8** *$L^k$ is in the big cell in $G$ and admits a factorization*

$$L^k = x_+^k x_-^k,\ x_+^k \in N_+,\ x_-^k \in N_-. \tag{4.5.11}$$

The elements $x_\pm^k$ are computed recursively from the known quantities. By applying a similar transform to the r.h.s. of (4.5.9) we get

$$L^k = s^{k-1}\left(m_k^{-1}\overrightarrow{\prod_p} n_p v_0 s \left(\overleftarrow{\prod_p} n_p^{-1}\right) s(m_k)\right) = \tag{4.5.12}$$

$$s^{k-1}\left(\overrightarrow{\prod_{p\geq k}} n_p v_0\right) s^k \left(\overleftarrow{\prod_{p\geq k+1}} n_p^{-1}\right) s^k\left(n_k^{-1}\right).$$

Comparison of (4.5.12) and (4.5.11) yields $x_-^k = s^k\left(n_k^{-1}\right)$. Hence $n_k = s^{-k}\left(x_-^k\right)^{-1}$, which concludes the induction.

Finally observe that by construction the map $\pi_q : N_+ s N_+ \rightarrow N'_+ s$ is a morphism of varieties.

**Corollary 4.5.9** *The space $\pi(\mu_{N_+}^{-1}(u))$ is a subvariety in $N'_+ s$. The algebra $\mathcal{F}(B_+)^{\mathcal{F}(N_-)}$ is isomorphic to the algebra of regular functions on $\pi(\mu_{N_+}^{-1}(u))$.*

*Proof.* First observe that by construction $\pi(\mu_{N_+}^{-1}(u)) \cong \pi_q(q(\mu_{N_+}^{-1}(u)))$ is a subvariety in $N'_+ s$. In particular, $\pi(\mu_{N_+}^{-1}(u))$ is a smooth manifold. Hence by Remark 4.2.6 the map

$$C^\infty(\pi(\mu_{N_+}^{-1}(u))) \rightarrow C^\infty(\mu_{N_+}^{-1}(u))^{C^\infty(N_-)},\quad \psi \mapsto \pi^*\psi$$

is an isomorphism.

Now observe that by construction the map $\pi : \mu_{N_+}^{-1}(u) \rightarrow \pi(\mu_{N_+}^{-1}(u))$ is a morphism of varieties. Therefore if $\psi \in \mathcal{F}(\pi(\mu_{N_+}^{-1}(u)))$ then $\pi^*\psi$ is a

regular function on $\mu_{N_+}^{-1}(u)$. Conversely, suppose that $\varphi \in \mathcal{F}(\mu_{N_+}^{-1}(u)) \cap C^\infty(\mu_{N_+}^{-1}(u))^{C^\infty(N_-)}$. Note that $\pi(\mu_{N_+}^{-1}(u))$ may be regarded as a subvariety in $\mu_{N_+}^{-1}(u)$ (see Remark 4.2.5). Then the restriction of $\varphi$ to $\pi(\mu_{N_+}^{-1}(u)) \subset \mu_{N_+}^{-1}(u)$ is a regular function. Therefore the map

$$\mathcal{F}(\pi(\mu_{N_+}^{-1}(u))) \to \mathcal{F}(\mu_{N_+}^{-1}(u)) \cap C^\infty(\mu_{N_+}^{-1}(u))^{C^\infty(N_-)}, \quad \psi \mapsto \pi^*\psi$$

is an isomorphism.

Finally observe that by Lemma 4.4.1 the algebra $\mathcal{F}(\mu_{N_+}^{-1}(u)) \cap C^\infty(\mu_{N_+}^{-1}(u))^{C^\infty(N_-)}$ is isomorphic to $\mathcal{F}(B_+)^{\mathcal{F}(N_-)}$. This completes the proof.

Theorem $\mathrm{C}_q$ is a group counterpart of Theorem C. Moreover the space $N'_+s$ naturally appears in the study of regular elements in $G$. Recall that an element of $G$ is called regular if its centralizer in $G$ is of minimal possible dimension. Let $R$ be the set of regular elements in $G$. Clearly, $R$ is stable under the action of $G$ on itself by conjugations and in fact $R$ is the union of all $G$ orbits in $G$ of maximal dimension. A function $\psi$ on $G$ is called a class function if $f(x) = f(y)$ whenever $x$ and $y$ are conjugate points of definition of $\psi$. We denote by $\mathcal{F}^G(G)$ the algebra of regular class functions on $G$.

**Theorem $\mathbf{D_q}$ ([29], Theorems 1.4 and 6.1)** *Let $G$ be a complex connected simply connected simple algebraic group. Then The space $N'_+s$ is contained in $R$ and is a cross–section for the action of $G$ on $R$. That is every $G$–orbit in $G$ of maximal dimension intersects $N'_+s$ in one and only one point. The algebra of regular class functions on $G$ is freely generated as a commutative algebra over $\mathbb{C}$ by the characters of fundamental representations of $G$, $\chi_1, \ldots, \chi_l$. Moreover, $N'_+s$ is an algebraic variety, and the algebra of regular functions on $N'_+s$ is freely generated as a commutative algebra over $\mathbb{C}$ by the restrictions of the characters $\chi_1, \ldots, \chi_l$ to $N'_+s$ .*

**Theorem $\mathbf{E_q}$** *For any $\psi \in \mathcal{F}^G(G)$ one has $\rho_{\chi^s}(p^*\psi) \in \mathcal{F}(B_+)^{\mathcal{F}(N_-)}$. Furthermore the map*

$$\mathcal{F}^G(G) \to \mathcal{F}(B_+)^{\mathcal{F}(N_-)}, \psi \mapsto \rho_{\chi^s}(p^*\psi) \tag{4.5.13}$$

*is an algebra isomorphism. In particular,*

$$\mathcal{F}(B_+)^{\mathcal{F}(N_-)} = \mathbb{C}[\rho_{\chi^s}(p^*\chi_1), \ldots, \rho_{\chi^s}(p^*\chi_l)]$$

*is a polynomial algebra in $l$ generators.*

*Proof.* Let $\psi$ be an element of $\mathcal{F}^G(G)$. The restriction of $\psi$ to the subvariety $\pi(\mu_{N_+}^{-1}(u)) \cong \pi_q(q(\mu_{N_+}^{-1}(u))) \subset N'_+s \subset G$ is a regular function. Using the isomorphism $\mathcal{F}(\pi(\mu_{N_+}^{-1}(u))) \cong \mathcal{F}(B_+)^{\mathcal{F}(N_-)}$ (see Corollary 4.5.9) this restriction may be identified with $\rho_{\chi^s}(q^*\psi) \in \mathcal{F}(B_+)^{\mathcal{F}(N_-)}$. By Theorem $\mathrm{D}_q$ the algebra $\mathcal{F}(N'_+s)$ is freely generated as a commutative algebra over $\mathbb{C}$ by the restrictions of the fundamental characters $\chi_1, \ldots, \chi_l$. Since $\pi(\mu_{N_+}^{-1}(u))$ is a subvariety in $N'_+s$ the algebra $\mathcal{F}(\pi(\mu_{N_+}^{-1}(u)))$ is generated by the restrictions of the fundamental characters $\chi_1, \ldots, \chi_l$. Therefore the map (4.5.13) is surjective. We have to prove that it is injective.

Let $\chi_i$ be a fundamental character. Consider the restriction of the function $\rho_{\chi^s}(q^*\chi_i)$ to the subspace in $\mu_{N_+}^{-1}(u)$ formed by elements (see (4.5.1)):

$$(h_+, s(h_+)u), \ h_+ \in H.$$

Then $\rho_{\chi^s}(q^*\chi_i)(h_+, s(h_+)u) = \chi_i(s(h_+)uh_+^{-1})$. Since $\chi_i$ is a character we have $\chi_i(s(h_+)uh_+^{-1}) = \chi_i(h_+^{-1}s(h_+)u)$. The element $u$ is unipotent, and hence $\chi_i(h_+^{-1}s(h_+)u) = \chi_i(h_+^{-1}s(h_+))$. Now recall that the restrictions of the fundamental characters to the Cartan subgroup are algebraically independent (they are given by the well–known Weyl formula). Therefore (4.5.13) is an isomorphism. This completes the proof.

*Proof of Theorem $B_q$.* Let $p : \mathcal{F}_h(G^*) \to \mathcal{F}(G^*)$ be the map defined in Proposition 4.3.3. Let $W_h^{Rep}(B_+)$ be the subalgebra in $W_h(B_+)$ topologically generated by the elements $C_{V_i}^{\rho_{\chi_h^s}} = \rho_{\chi_h^s}(C_{V_i}), \ i = 1, \ldots, l$. From the the definition of the elements $C_{V_i}^{\rho_{\chi_h^s}}$ it follows that $p(C_{V_i}^{\rho_{\chi_h^s}}) = \rho_{\chi^s}(p^*\chi_i)$. Therefore by Theorem $\mathrm{E}_q$ $p(W_h^{Rep}(B_+)) = \mathcal{F}(B_+)^{\mathcal{F}(N_-)}$, and $W_h^{Rep}(B_+)$ is freely generated as a commutative topological algebra over $\mathbb{C}[[h]]$ by the elements $C_{V_i}^{\rho_{\chi_h^s}} = \rho_{\chi_h^s}(C_{V_i}), \ i = 1, \ldots, l$.

On the other hand using the definitions of the algebras $\mathcal{F}_h(B_+)^{\mathcal{F}_h(N_-)}$ and $\mathcal{F}(B_+)^{\mathcal{F}(N_-)}$ it is easy to see that $p(\mathcal{F}_h(B_+)^{\mathcal{F}_h(N_-)}) = \mathcal{F}(B_+)^{\mathcal{F}(N_-)}$. We shall prove that $W_h^{Rep}(B_+)$ is isomorphic to $\mathcal{F}_h(B_+)^{\mathcal{F}_h(N_-)}$.

Let $I \in \mathcal{F}_h(B_+)^{\mathcal{F}_h(N_-)}$ be an invariant element. Then $p(I) \in \mathcal{F}(B_+)^{\mathcal{F}(N_-)}$, and hence one can find an element $K_0 \in W_h^{Rep}(B_+)$ such that $I - K_0 = hI_1, \ I_1 \in \mathcal{F}_h(B_+)^{\mathcal{F}_h(N_-)}$. Applying the same procedure to $I_1$ one can find elements $K_1 \in W_h^{Rep}(B_+), \ I_2 \in \mathcal{F}_h(B_+)^{\mathcal{F}_h(N_-)}$ such that $I_1 - K_1 = hI_2$, i.e. $I - K_0 - hK_1 = 0 \pmod{h^2}$. We can continue this process. Finally we obtain an infinite sequence of elements $K_i \in W_h^{Rep}(B_+)$ such that $I - \sum_{i=0}^{p} h^p K_p = 0 \pmod{h^{p+1}}$. Since the space $\mathcal{F}_h(B_+)$ is complete in the

$h$–adic topology the series $\sum_{i=0}^{\infty} h^p K_p \in W_h^{Rep}(B_+)$ converges to $I$. Therefore $I \in W_h^{Rep}(B_+)$, and hence $\mathcal{F}_h(B_+)^{\mathcal{F}_h(N_-)}$ is isomorphic to $W_h^{Rep}(B_+)$.

We also have the following inclusions:

$$W_h^{Rep}(B_+) \subseteq W_h(B_+) \subseteq \mathcal{F}_h(B_+)^{\mathcal{F}_h(N_-)} \cong W_h^{Rep}(B_+).$$

Therefore $W_h^{Rep}(B_+)$ coincides with $W_h(B_+)$. This proves part (ii) of Theorem $\mathrm{A}_q$ and Theorem $\mathrm{B}_q$.

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